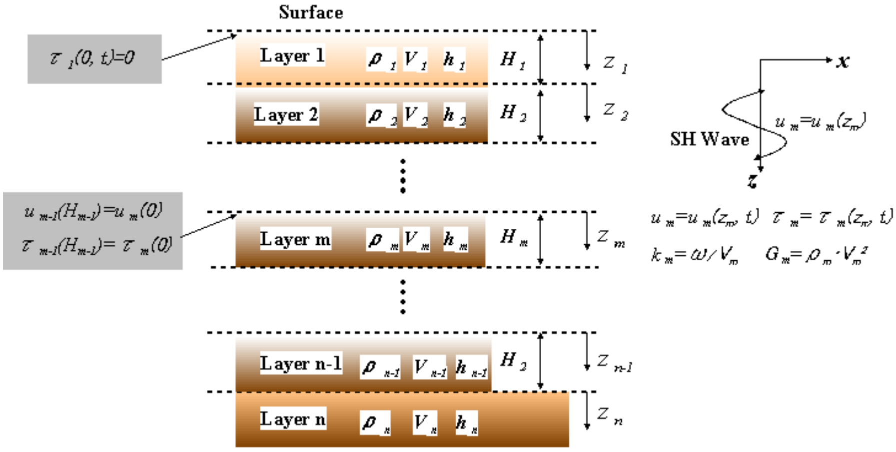
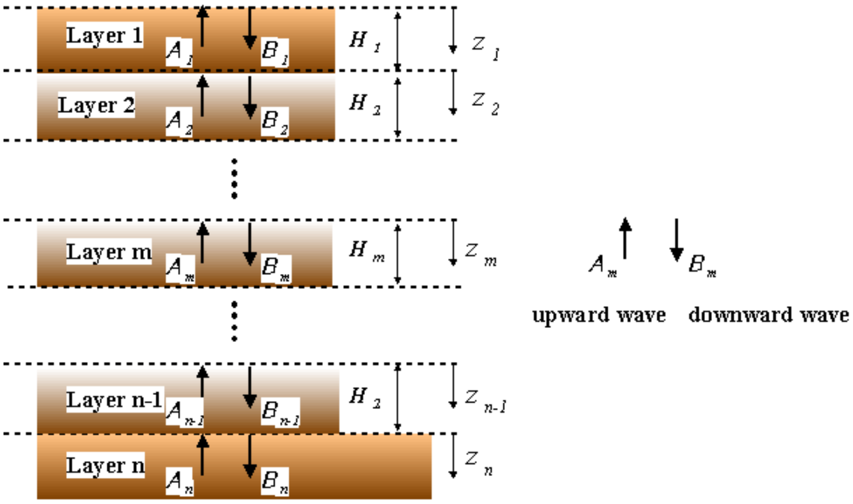
Multi-reflection theory for one-dimensional wave in multi-layers system



Multi-Layer System and Coordinate System



Reflection and Refraction

$$\rho\frac{\partial^2 u}{\partial t^2}=G\frac{\partial}{\partial z}(\gamma)+c\frac{\partial}{\partial z}\left(\frac{\partial\gamma}{\partial t}\right)=G\frac{\partial^2 u}{\partial z^2}+\eta\frac{\partial^3 u}{\partial z^2\partial t}$$: *refer to Navier-Stokes Equation for viscous fluid* (A)

$$\eta=\frac{G}{\omega}2h$$: h is damping factor. (B)

$$u(z,t)=\exp\left[i\omega\left(t-\frac{z}{V_s^*}\right)\right]$$, here, V_s^* indicates shear velocity with account for viscosity (C)

Substituting (C) into (A), we can obtain,

$$\rho\left(V_s^*\right)^2=G+i\eta\omega$$ (D)

$$V_s^*=\sqrt{G+i\eta\omega}=V_s\sqrt{G+i\eta\omega/G}$$: complex shear velocity (E)

where, $\eta\omega/G$ is usually small so that it could be negligible, so we apply Taylor's expansion to (E) and then (E) reduce to

$$\frac{1}{V_s^*}\approx\frac{1}{V_s}\left(1-\frac{1}{2}i\frac{\omega\eta}{G}\right)$$ (F).

Substituting (F) into (C), we can obtain,

$$u(z,t)=\exp\left[i\omega\left(t-\frac{z}{V_s}\right)\right]\exp\left[-\frac{\eta\omega^2 z}{2V_sG}\right]$$ (G)

This solution means that wave is attenuated while propagating with velocity of V_s and the damping is exponential term which is proportional to η.

Otherwise, logarithm damping ratio $\delta=\ln\left(\frac{a_1}{a_2}\right)=\ln\left(\frac{a_2}{a_3}\right)=....=2\pi h$. => eq.(B)

Therefore, displacement can be described as

$$u(z,t)=\exp\left[i\omega\left(t-\frac{z}{V_s}\right)\right]\exp\left[-\frac{2\pi h\cdot z}{\lambda G}\right]$$ ($\lambda=\frac{2\pi V_s}{\omega}$: wave length) (H)

When we use G^*, (A) can be rewritten by the following equation.

$$\rho\frac{\partial^2 u}{\partial t^2}=G^*\frac{\partial^2 u}{\partial z^2}$$ (I)

$$G^*=G(1+i2h)$$: complex shear modulus (J)

(one of empirical relations: $h\approx 0.017+0.0002V_s$)

1) Upward wave and Downward wave with only SH component in Horizontal Layers

Displacement function in the m-th layer

$$u_m(z_m,t)=A_m\exp[i(k_m z_m+\omega t)]+B_m\exp[i(k_m z_m-\omega t)] \quad (4.1)$$

$$k_m=\frac{\omega}{V_m}=\frac{2\pi}{\lambda_m}\text{: wave number } (V_m=f\lambda_m=\frac{\omega}{2\pi}\lambda_m) \quad (4.2)$$

$$V=V^{\mathrm{Re}}\sqrt{1+ic\omega}\text{: complex velocity} \quad (4.3)$$

$$G\approx G^{\mathrm{Re}}(1+i2h)\text{: complex shear modulus} \quad (4.4)$$

Shear stress – shear strain described by displacement fields

$$\gamma=\frac{\partial u_m}{\partial z_m}=ik_m\left[A_m\exp\{i(k_m z_m+\omega t)\}-B_m\exp\{-i(k_m z_m-\omega t)\}\right] \quad (4.5)$$

$$\tau_m(z_m,t)=iG_m k_m\left[A_m\exp\{i(k_m z_m+\omega t)\}-B_m\exp\{-i(k_m z_m-\omega t)\}\right] \quad (4.6)$$

Matrix description in m-th layer for amplitude

$$\begin{Bmatrix}u_m(z_m)\\ \tau_m(z_m)\end{Bmatrix}=\begin{bmatrix}\exp(ik_m z_m) & \exp(-ik_m z_m)\\ iG_m k_m\exp(ik_m z_m) & -iG_m k_m\exp(-ik_m z_m)\end{bmatrix}\begin{Bmatrix}A_m\\ B_m\end{Bmatrix} \quad (4.7)$$

at $z_m=0$

$$\begin{Bmatrix}u_m(0)\\ \tau_m(0)\end{Bmatrix}=\begin{bmatrix}1 & 1\\ iG_m k_m & -iG_m k_m\end{bmatrix}\begin{Bmatrix}A_m\\ B_m\end{Bmatrix} \quad (4.8)$$

at $z_m=z_m$

$$\therefore\quad \begin{Bmatrix}u_m(z_m)\\ \tau_m(z_m)\end{Bmatrix}=\begin{bmatrix}\cos k_m z_m & \dfrac{\sin k_m z_m}{G_m k_m}\\ -G_m k_m\sin\sin k_m z_m & \cos k_m z_m\end{bmatrix}\begin{Bmatrix}u_m(0)\\ \tau_m(0)\end{Bmatrix} \quad (4.9)$$

at $z_m=H_m$

$$\therefore\quad \begin{Bmatrix}u_m(H_m)\\ \tau_m(H_m)\end{Bmatrix}=[S_m]\begin{Bmatrix}u_m(0)\\ \tau_m(0)\end{Bmatrix} \quad (4.10)$$

$$[S_m]=\begin{bmatrix}\cos k_m H_m & \dfrac{\sin k_m H_m}{G_m k_m}\\ -G_m k_m\sin k_m H_m & \cos\sin k_m H_m\end{bmatrix}\begin{Bmatrix}u_m(0)\\ \tau_m(0)\end{Bmatrix} \quad (4.11)$$

Boundary condition at the interface between (m-1)th and m-th layers

$$\textbf{at}\ \ z_{m-1}=H_{m-1}\ \ \textbf{and}\ \ z_m=0\ \begin{Bmatrix}u_m(0)\\ \tau_m(0)\end{Bmatrix}=\begin{Bmatrix}u_{m-1}(H_{m-1})\\ \tau_{-1}(H_{m-1})\end{Bmatrix} \quad (4.12)$$

$$\begin{Bmatrix}u_m(0)\\ \tau_m(0)\end{Bmatrix}=[S_{m-1}]\begin{Bmatrix}u_{m-1}(0)\\ \tau_{m-1}(0)\end{Bmatrix} \quad (4.13)$$

Recurrence formulation for amplitude in m-th layer assessed by that of 1st layer

$$\begin{Bmatrix} u_m(0) \\ \tau_m(0) \end{Bmatrix} = [R_{m-1}] \begin{Bmatrix} u_1(0) \\ \tau_1(0) \end{Bmatrix} \tag{4.14}$$

$$\left(\begin{Bmatrix} u_m(0) \\ \tau_m(0) \end{Bmatrix} = [S_{m-1}] \begin{Bmatrix} u_{m-1}(0) \\ \tau_{m-1}(0) \end{Bmatrix} = [S_{m-1}][S_{m-2}] \begin{Bmatrix} u_{m-2}(0) \\ \tau_{m-2}(0) \end{Bmatrix} = \ldots\ldots \right)$$

where,

$$[R_{m-1}] = [S_{m-1}][S_{m-2}] \cdots\cdots [S_1] \tag{4.15}$$

at the ground surface: $z_1 = 0$

$$\tau_1(0) = 0 \tag{4.16}$$

Therefore,

$$\therefore \quad \begin{Bmatrix} u_m(0) \\ \tau_m(0) \end{Bmatrix} = [R_{m-1}] \begin{Bmatrix} u_1(0) \\ 0 \end{Bmatrix} \tag{4.17}$$

The above results are summarized as follows:

$$\left. \begin{aligned} \begin{Bmatrix} u_m(0) \\ \tau_m(0) \end{Bmatrix} &= [R_{m-1}] \begin{Bmatrix} u_1(0) \\ 0 \end{Bmatrix} \\ \begin{Bmatrix} u_m(z_m) \\ \tau_m(z_m) \end{Bmatrix} &= \begin{bmatrix} \cos k_m z_m & \dfrac{\sin k_m z_m}{G_m k_m} \\ -G_m k_m \sin \sin k_m z_m & \cos k_m z_m \end{bmatrix} \begin{Bmatrix} u_m(0) \\ \tau_m(0) \end{Bmatrix} \end{aligned} \right\} \tag{4.18}$$

2) Frequency Response Function *（周波数応答関数）*

From Eq.(4.1),

$$\begin{Bmatrix} A_n \\ B_n \end{Bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & -\dfrac{i}{G_n k_n} \\ 1 & \dfrac{1}{G_n k_n} \end{bmatrix} \begin{Bmatrix} u_n(0) \\ \tau_n(0) \end{Bmatrix} \tag{4.19}$$

$$\left.\begin{aligned} u_n(0) &= \mathbf{R}_{n-1}(1,1)\, u_1(0) \\ \tau_n(0) &= \mathbf{R}_{n-1}(2,1)\, u_1(0) \end{aligned}\right\} \tag{4.20}$$

$$A_n = \frac{u_1(0)}{2}\left[R_{n-1}(1,1) - i\,\frac{R_{n-1}(2,1)}{G_n k_n} \right] \tag{4.21}$$

$$U_1(\omega) = \frac{2}{\sqrt{\left[R_{n-1}(1,1)\right]^2 + \left[R_{n-1}(2,1)/G_n k_n\right]^2}} \tag{4.22}$$

3) Amplification Factor (増幅率)

From Eq.(4.13) and Eq.(4.14)

$$\begin{Bmatrix} A_m \\ B_m \end{Bmatrix} = [T_{m-1}] \begin{Bmatrix} A_{m-1} \\ B_{m-1} \end{Bmatrix} \tag{4.23}$$

$$[T_{m-1}] = \begin{bmatrix} \frac{1}{2}(1+\alpha_{m-1})\exp(ik_{m-1}H_{m-1}) & \frac{1}{2}(1-\alpha_{m-1})\exp(-ik_{m-1}H_{m-1}) \\ \frac{1}{2}(1-\alpha_{m-1})\exp(ik_{m-1}H_{m-1}) & \frac{1}{2}(1+\alpha_{m-1})\exp(-ik_{m-1}H_{m-1}) \end{bmatrix}$$

(4.24)

where

$$\boxed{\alpha_{m-1} = \frac{G_{m-1}k_{m-1}}{G_m k_m} = \frac{\rho_{m-1}V_{m-1}}{\rho_m V_m}} \tag{4.25}$$

$\rho_m V_m$ ***is wave impedance(波動インピーダンス)***

<u>α ***is impedance ratio***</u>

From Eq.(4.8) and Eq.(4.17), we can obtain the following relation,

$$\begin{Bmatrix} u_m(0) \\ \tau_m(0) \end{Bmatrix} = \begin{bmatrix} 1 & 1 \\ iG_m k_m & -iG_m k_m \end{bmatrix} \begin{Bmatrix} A_m \\ B_m \end{Bmatrix} \tag{4.7'}$$

$$\tau_1(0) = 0 \tag{4.16}$$

the following relation is obtained

$A_1 = B_1$ (from Eq.(4.7)' and Eq.(4.16)) (4.26)

$u_1(0) = 2A_1$ (from Eq.(4.7)' and Eq.(4.26)) (4.27)

Amplitude in m-th layer assessed by that of 1st layer

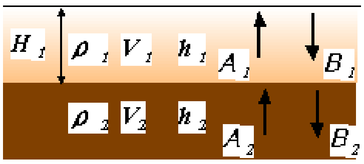
$$\begin{Bmatrix} A_m \\ B_m \end{Bmatrix} = [Q_{m-1}] \begin{Bmatrix} A_1 \\ A_1 \end{Bmatrix} \tag{4.28}$$

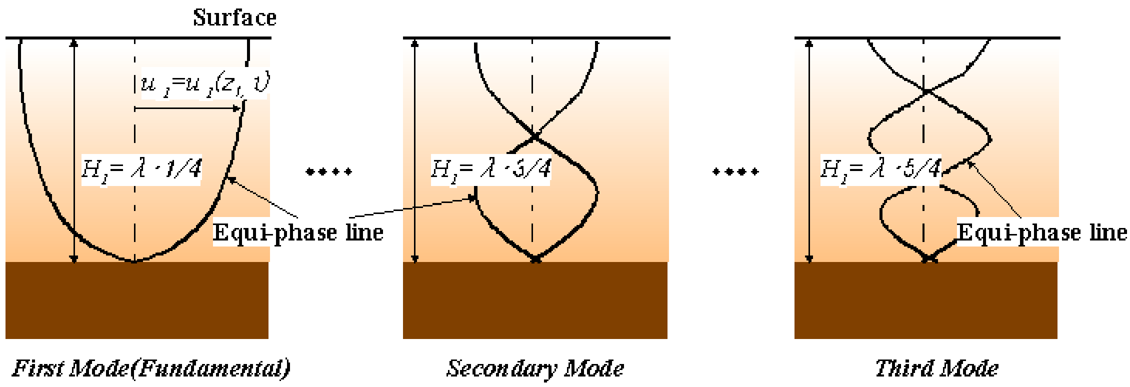
where,

$$[Q_{m-1}] = [T_{m-1}][T_{m-2}]\cdots\cdots[T_1] \tag{4.29}$$

4) Fundamental consideration in two-layer system

Resonance and sympathetic vibration in tow-layer system





Geometries in resonance of two layer system

For two layers,

[Amplification]

$$A_1 = B_1 = \frac{1}{\cos(k_1 H_1) + i\alpha_1 \sin(k_1 H_1)} A_2 \qquad \text{(a)}$$

$$B_2 = \frac{\cos(k_1 H_1) - i\alpha_1 \sin(k_1 H_1)}{\cos(k_1 H_1) + i\alpha_1 \sin(k_1 H_1)} A_2 \qquad \text{(b)}$$

[Natural Period(Frequency)]

$$\cos k_1 H_1 = 0$$

$$\omega_n = 2\pi f_n = 2\pi \frac{V_1}{\lambda_n} = 2\pi V_1 \frac{1}{\left(H_1 \frac{4}{(2l-1)} \right)} = (2l-1) \frac{\pi V_1}{2H_1} \quad (l = 1, 2, \ldots\ldots)$$

$$H_1 = \frac{\lambda_1}{4}(2l-1), \quad V_1 = f\lambda_1 = f\frac{4}{(2l-1)} H_1$$

$$T_n = \frac{2\pi}{\omega_n} = \frac{1}{f} = V_1 \frac{(2l-1)}{4H_1} \qquad \text{(c)}$$

[Energy]

$$E = \frac{1}{2}\rho \cdot V \cdot \omega^2 \cdot |A|^2 \qquad \text{(d)}$$

Propagation energy per unit time

kinematics

$$(E_K)_{\max} = \left[\frac{1}{2} m \cdot V^2\right] = \frac{1}{2}\rho V_s \cdot (V)^2 = \frac{1}{2}\rho V \cdot (\omega |A|)^2$$

potential

$$(E_U)_{\max} = \left(\frac{1}{2} G \cdot \gamma_{\max}^2\right) \cdot V_s = \left(\frac{1}{2}\rho V_s^2 \cdot (k|A|)^2\right) \cdot V = \left(\frac{1}{2}\rho \cancel{V_s^2} \cdot \frac{\omega^2}{\cancel{V_s^2}} |A|^2\right) \cdot V_s$$

If a wave propagates from hard material to soft material (V_s decreases), the amplitude increases under constant energy.

$$\frac{E_T}{E_0} = \frac{\alpha_1}{\cos^2(k_1 H_1) + \alpha_1^2 \sin^2(k_1 H_1)} \qquad \text{(e)}$$

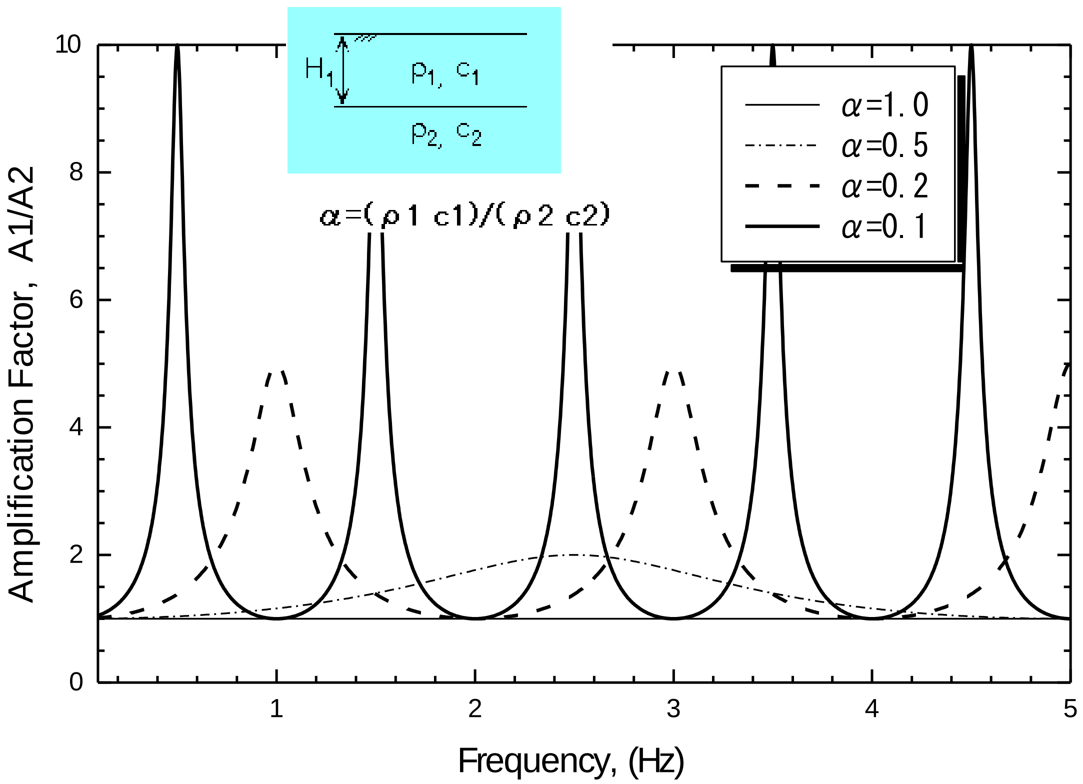
$$\frac{E_T}{E_0} = 1$$

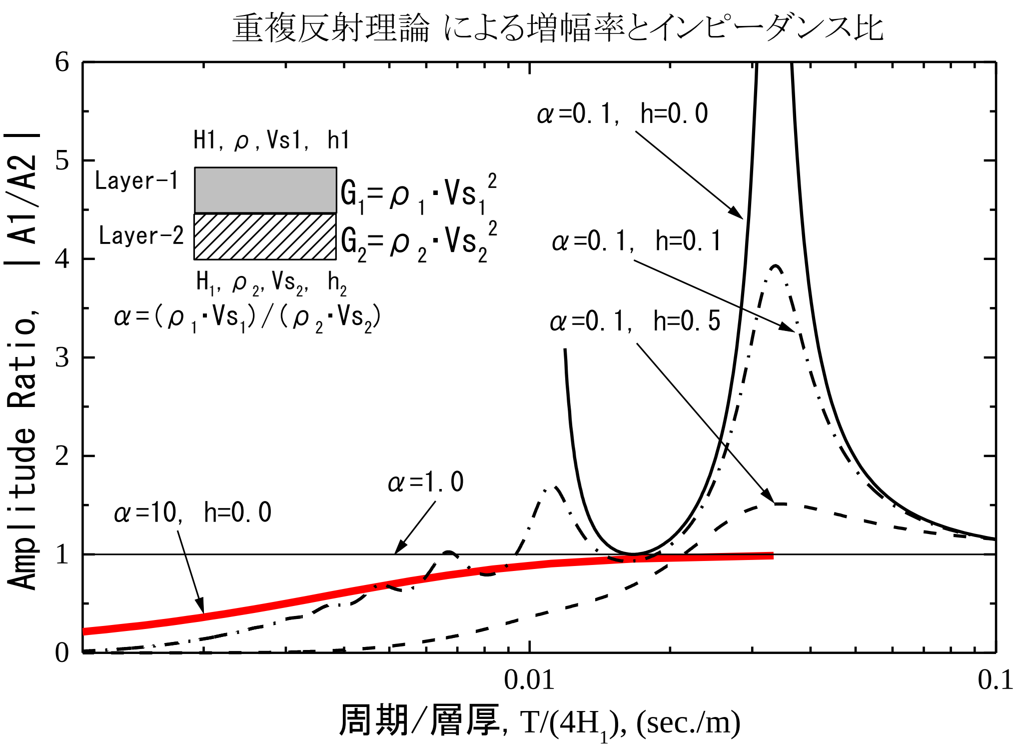
For multi-layers,

[Natural Period(Frequency)] ***Simple approximations***

$$\text{(i)} \quad T = \frac{\lambda_1}{V_1} + \cdots + \frac{\lambda_n}{V_n} = \frac{4H_1}{V_1} + \cdots + \frac{4H_n}{V_n} = \sum_{j=1}^{n} \frac{4H_j}{V_j}$$

$$\text{(ii)} \quad T = \frac{4\sum_{j=1}^{n} H_j}{V_{ave}}, \qquad V_{ave} = \frac{\sum_{j=1}^{n} V_j H_j}{\sum_{j=1}^{n} H_j}$$


H1
ρ1, c1
ρ2, c2
α=(ρ1 c1)/(ρ2 c2)
α=1.0
α=0.5
α=0.2
α=0.1
Amplification Factor, A1/A2
Frequency, (Hz)
0
2
4
6
8
10
1
2
3
4
5


重複反射理論 による増幅率とインピーダンス比
Amplitude Ratio, |A1/A2|
H1, ρ, Vs1, h1
Layer-1
Layer-2
G1=ρ1・Vs1^2
G2=ρ2・Vs2^2
H1, ρ2, Vs2, h2
α=(ρ1・Vs1)/(ρ2・Vs2)
α=0.1, h=0.0
α=0.1, h=0.1
α=0.1, h=0.5
α=1.0
α=10, h=0.0
周期/層厚, T/(4H1), (sec./m)
0
1
2
3
4
5
6
0.01
0.1

Examples for seismic isolation effect