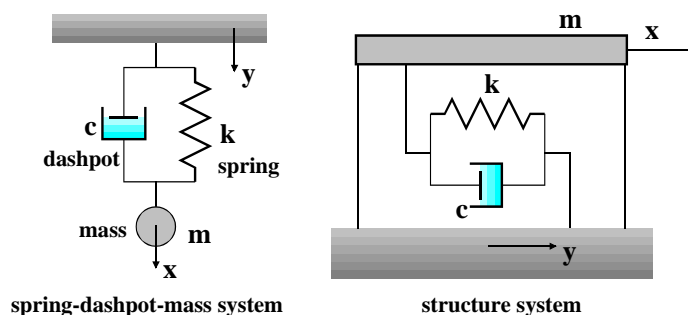


1 質点線形系の振動解析

Numerical Analysis Method for a Mass with Linear-System in 1-D



Linear Vibration System

- 1 質点系の振動の数学的厳密解(Mathematical Exact Solution)
- 応答の数値解法(Numerical Solutions) : 直接積分法(Direct Integration Method)
- 地震応答スペクトル (Earthquake Response Spectrum)

運動方程式 :

$$m\ddot{x}(t) + c\dot{x}(t) + kx(t) = f(t) = -m\ddot{y}(t) \quad (1.1)$$

(慣性力 : 相対加速度) + (減衰制振力 : 相対速度) + (復元力 : 相対変位) = (外力)

Inertia force viscous force recover force

釣合い式 :

$$-m(\ddot{x}(t) + \ddot{y}(t)) - c\dot{x}(t) - kx(t) = 0 \quad (1.2)$$

(慣性力 : 絶対加速度) + (減衰制振力 : 相対速度) + (復元力 : 相対変位) = 0

m : mass [kg]

k : spring coefficient [kg/sec²]

c : viscous damping coefficient [kg/sec]

$f(t)$: external force applied to the mass

$\ddot{y}(t)$: $f(t)$ is replaced with acceleration of the ceiling and the ground $-m\ddot{y}(t)$ according to D'Alembert's law.

$\ddot{x}(t)$, $\dot{x}(t)$, $x(t)$: relative acceleration [m/sec²], velocity [m/sec], and displacement [m] with respect to the ceiling or the ground, respectively

$x(t) + y(t)$: absolute displacement

$m\ddot{x}(t)$: inertia force

1. Mathematical Exact Solutions: Analytical results

1.1. 自由振動 (Free Vibration) :

$$f(t) \equiv 0, \quad \ddot{y}(t) \equiv 0 \quad (1.3)$$

Linear and homogeneous differential equation

solution = general solution

$$x(t) = x_g(t) \quad (1.4)$$

Characteristic Equation using differential operator

$$D^2 x_g + 2\gamma D x_g + \omega_o^2 x_g = 0 \quad (1.5)$$

D: differential operator with respect to time differential (D = d / dt): $\dot{x}_g = D x_g$

and $\ddot{x}_g = D^2 x_g$.

$$\gamma = \frac{c}{2m}, \quad \omega_o^2 = \frac{k}{m} \quad (1.6)$$

γ : viscosity normalized by mass[1/sec]

ω_o : referential angular frequency[1/sec] / natural circle frequency

$$T_o = \frac{1}{f_o} = \frac{2\pi}{\omega_o} = 2\pi \sqrt{\frac{k}{m}} \quad (1.7)$$

T_o : referential vibration period[sec] / natural period

f_o : referential vibration frequency[1/sec] / natural frequency

$$D x_g = \left(\gamma \pm \sqrt{\gamma^2 - \omega_o^2} \right) x_g \quad (1.8)$$

Vibration mode of the system clearly depends on the relative magnitude of viscous damping:

$$\begin{cases} \gamma^2 - \omega_o^2 < 0 & \left(c < 2\sqrt{mk} \right) & \text{normally damped vibration} \\ \gamma^2 - \omega_o^2 = 0 & \left(c = 2\sqrt{mk} \right) & \text{critically damped vibration : } c_{cr} = 2\sqrt{mk} \\ \gamma^2 - \omega_o^2 > 0 & \left(c > 2\sqrt{mk} \right) & \text{over damped vibration} \end{cases}$$

(1.9)

a) *Normally damped vibration*

parameters

$$c \leq c_{cr} = 2\sqrt{mk} , \quad (2.10)$$

$$D = (-\gamma \pm i\omega), \quad (2.11)$$

$$\omega^2 = \omega_o^2 - \gamma^2, \quad (2.12)$$

period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_o} \frac{1}{\sqrt{1-h^2}} = \frac{T_o}{\sqrt{1-h^2}} \quad (2.13)$$

motions

$$\begin{cases} x_g(t) = e^{-\gamma t} [A_g \cos \omega t + B_g \sin \omega t] \\ \dot{x}_g(t) = e^{-\gamma t} \left[\{-\gamma A_g + \omega B_g\} \cos \omega t + \{-\gamma B_g - \omega A_g\} \sin \omega t \right] \\ \ddot{x}_g(t) = e^{-\gamma t} \left[\{(\gamma^2 - \omega^2)A_g - 2\gamma\omega B_g\} \cos \omega t + \{(\gamma^2 - \omega^2)B_g + 2\gamma\omega A_g\} \sin \omega t \right] \end{cases} \quad (2.14)$$

$$\text{initial conditions} \quad \begin{cases} x_g(0) = A_g \\ \dot{x}_g(0) = -\gamma A_g + \omega B_g \end{cases} \quad (2.15)$$

$$\text{therefore,} \quad \begin{cases} A_g = x_g(0) \\ B_g = \{\dot{x}_g(0) + \gamma x_g(0)\} / \omega \end{cases} \quad (2.16)$$

complex descriptions

$$\begin{cases} x_g(t) = \text{Real} [C_g \exp\{(-\lambda - i\omega)t\}] \\ \dot{x}_g(t) = \text{Real} [C_g (-\lambda - i\omega) \exp\{(-\lambda - i\omega)t\}] \\ \ddot{x}_g(t) = \text{Real} [C_g (-\lambda - i\omega)^2 \exp\{(-\lambda - i\omega)t\}] \end{cases} \quad (2.17)$$

damping

Viscous damping coefficient

c

$$c_{cr} = 2\sqrt{mk} \quad (2.18)$$

Viscosity normalized by mass

$$\gamma = \frac{c}{2m}, \tag{2.19}$$

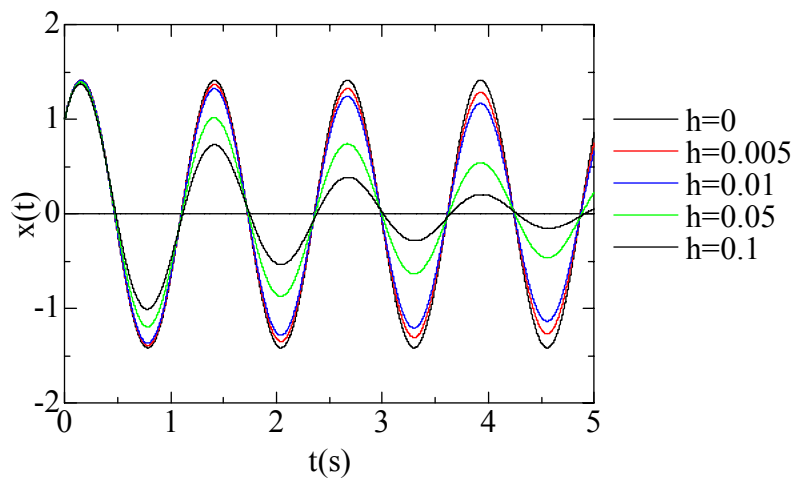
Logarithmic damping factor

$$D = \ln \frac{a_n}{a_{n+1}} \tag{2.20}$$

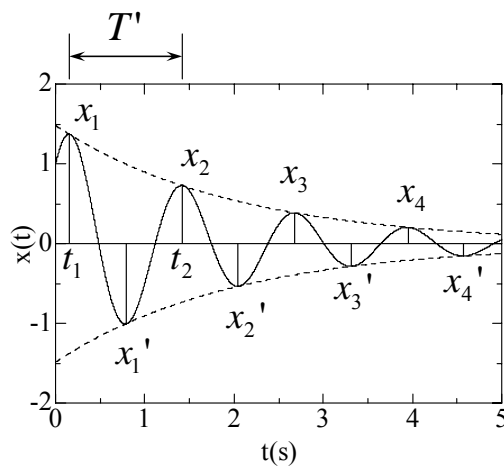
Damping factor

$$h = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}} = \frac{\gamma}{\omega_o} \tag{2.21}$$

$$D = \ln \frac{a_n}{a_{n+1}} = \ln \frac{x(t)}{x(t+T)} = \gamma T = \frac{2\pi h}{\sqrt{1-h^2}} \tag{2.22}$$



自由減衰振動



固有周期実測値(observed value of fundamental natural period)

<i>structure</i>	中低層 (H≦45m)	高層 (H < 45m)
S	$T_1=0.061 N[\text{階}]$	$T_1=0.079 N[\text{階}]$
SRC, RC	$T_1=0.054 N[\text{階}]$	$T_1=0.053 N[\text{階}]$

減衰定数実測値(observed value of damping factor)

鉄骨構造(Steel structure): $h=2\%$

鉄骨鉄筋コンクリート構造(Steel reinforced concrete structure): $h=3\%$

鉄筋コンクリート構造(Reinforced concrete structure) : $h=5\%$

土 : $h=0\sim 25\%$

Here, if $h \ll 1$, we can obtain,

$$h \ll 1, \quad D = 2\pi h \quad (2.23)$$

Consequently, we can estimate value of h by measuring D ,

$$h \ll 1, \quad h = \frac{D}{2\pi} \quad (2.24)$$

b) Critical damped vibration

parameters

$$c = c_{cr} = 2\sqrt{mk} , \quad (2.25)$$

$$D = -\gamma , \quad (2.26)$$

$$\omega^2 = \omega_o^2 - \gamma^2 = 0 , \quad (2.27)$$

period

$$T = \frac{2\pi}{\omega} \Rightarrow \infty \quad (2.28)$$

motions

$$\begin{cases} x_g(t) = e^{-\gamma t} [A_g + \gamma B_g t] \\ \dot{x}_g(t) = e^{-\gamma t} [(-\gamma A_g + \gamma B_g) + (-\gamma^2 B_g t)] \\ \ddot{x}_g(t) = e^{-\gamma t} [\gamma^2 A_g - 2\gamma^2 B_g + \gamma^3 B_g t] \end{cases} \quad (2.29)$$

$$\text{initial conditions} \quad \begin{cases} x_g(0) = A_g \\ \dot{x}_g(0) = -\gamma A_g + \gamma B_g \end{cases} \quad (2.30)$$

$$\text{therefore,} \quad \begin{cases} A_g = x_g(0) \\ B_g = \dot{x}_g(0)/\gamma + x_g(0) \end{cases} \quad (2.31)$$

c) Over damped vibration

parameters

$$c \geq c_{cr} = 2\sqrt{mk} , \quad (2.32)$$

$$D = (-\gamma \pm \omega) , \quad (2.33)$$

$$\omega^2 = -(\omega_o^2 - \gamma^2) , \quad (2.34)$$

period

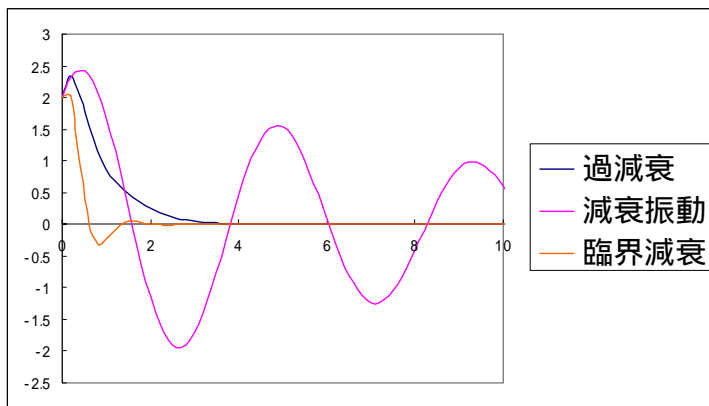
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_o} \frac{1}{\sqrt{h^2 - 1}} = \frac{T_o}{\sqrt{h^2 - 1}} \quad (2.35)$$

motions

$$\begin{cases} x_g(t) = e^{-\gamma t} [A_g \cosh \omega t + B_g \sinh \omega t] \\ \dot{x}_g(t) = e^{-\gamma t} \left[\{-\gamma A_g + \omega B_g\} \cosh \omega t + \{-\gamma B_g - \omega A_g\} \sinh \omega t \right] \\ \ddot{x}_g(t) = e^{-\gamma t} \left[\{(\gamma^2 - \omega^2)A_g - 2\gamma\omega B_g\} \cosh \omega t + \{(\gamma^2 - \omega^2)B_g + 2\gamma\omega A_g\} \sinh \omega t \right] \end{cases} \quad (2.36)$$

initial conditions $\begin{cases} x_g(0) = A_g \\ \dot{x}_g(0) = -\gamma A_g + \omega B_g \end{cases} \quad (2.37)$

therefore, $\begin{cases} A_g = x_g(0) \\ B_g = \{\dot{x}_g(0) + \gamma x_g(0)\} / \omega \end{cases} \quad (2.38)$



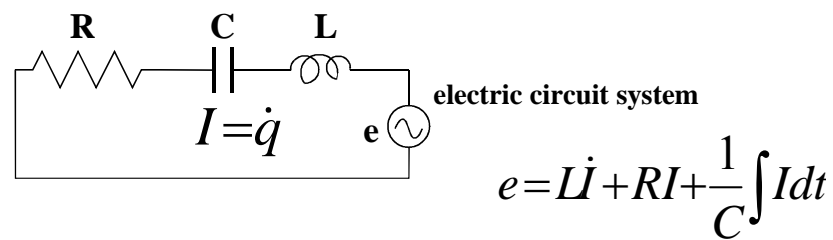
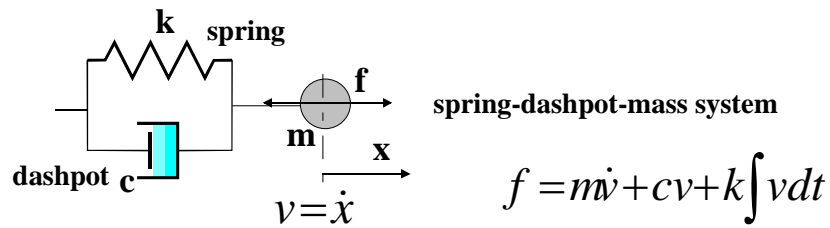
An example for vibration behaviours with different h : normally damped, critical damped and over damped vibrations

* Euler's formula

$$\cos x = \frac{1}{2}(e^{ix} + e^{-ix}), \quad \sin x = \frac{1}{2}(e^{ix} - e^{-ix})$$

$$e^{inx} = (\cos x + i \sin x)^n = (\cos nx + i \sin nx)$$

* mechanical vibration system and electric circuit system



* mechanical vibration system and a control system in control engineering: PID control

$$S_c = T_D K_p \frac{de}{dt} + K_p e + \frac{1}{T_I} K_p \int e dt$$

Derivation control + Proportion control + Integration control
PID

S_c : control signal

e : error = target value – current value

非線形解法 (Non-linear analysis of vibration by numerical method)

直接積分法(Direct Integration Method)

非線形運動方程式の数値積分の方法

㉔ 差分法

- 時間領域の数値積分

- Newmark の β 法： 実用的に最も用いられる

$$x(t + \Delta t) = x(t) + \Delta t \cdot \dot{x}(t) + \left(\frac{1}{2} - \beta\right)(\Delta t)^2 \ddot{x}(t) + \beta(\Delta t)^2 \cdot \ddot{x}(t + \Delta t) \quad (2.39)$$

$$\dot{x}(t + \Delta t) = \dot{x}(t) + (1 - \gamma)\Delta t \cdot \ddot{x}(t) + \gamma \cdot \Delta t \cdot \ddot{x}(t + \Delta t) \quad (2.40)$$

Newmark の β 法の特別な呼び名

手法	γ	β
線形加速度法(Linear Acceleration Method)	0.5	1/6
中点加速度法(Constant Average Acceleration Method)	0.5	1/4

- Wilson の θ 法

$$x(t + \Delta t) = x(t) + \Delta t \cdot \dot{x}(t) + \frac{1}{2}(\Delta t)^2 \ddot{x}(t) + \frac{(\Delta t)^2}{6\theta}(\ddot{x}(t + \theta\Delta t) - \ddot{x}(t)) \quad (2.41)$$

$$\dot{x}(t + \Delta t) = \dot{x}(t) + \Delta t \cdot \ddot{x}(t) + \frac{\Delta t}{2\theta}(\ddot{x}(t + \Delta t) - \ddot{x}(t)) \quad (2.42)$$

各種数値積分法の安定条件(stability condition of numerical integration)

手法	数値積分の安定条件	無条件安定条件 (unconditionally Stable)
Newmark の β 法	$\Delta t \leq \frac{2}{\omega_o \sqrt{1 - 4\beta}} \cdot 0.5$	$\beta \leq 1/4$
Wilson の θ 法	$\Delta t \leq \frac{2}{\omega_o \sqrt{1 - 4\beta}} \cdot 0.5$	$\theta \geq 1.37$
中央差分法	$\Delta t \leq \frac{2}{\omega}$	なし

ω_o はモデルの最も大きい円振動数

- 直接積分法

□) 運動方程式の離散化

$$\ddot{x}(t) + 2h\omega \dot{x}(t) + \omega^2 x(t) = -\ddot{y}(t) \quad (2.43)$$

$$\begin{cases} x_{t+\Delta t} = A_{11}x_t + A_{12}\dot{x}_t + B_{11}\ddot{y}_t + B_{12}\ddot{y}_{t+\Delta t} \\ \dot{x}_{t+\Delta t} = A_{21}x_t + A_{22}\dot{x}_t + B_{21}\ddot{y}_t + B_{22}\ddot{y}_{t+\Delta t} \end{cases} \quad (2.44)$$

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad [B] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \quad (2.45)$$

$$\begin{cases} x_{t+\Delta t} \\ \dot{x}_{t+\Delta t} \end{cases} = [A] \begin{cases} x_t \\ \dot{x}_t \end{cases} + [B] \begin{cases} \ddot{y}_t \\ \ddot{y}_{t+\Delta t} \end{cases} \quad (2.46)$$

$$\ddot{x}_{t+\Delta t} + 2h\omega \dot{x}_{t+\Delta t} + \omega^2 x_{t+\Delta t} = -\ddot{y}_{t+\Delta t} \quad (2.47)$$

$$\ddot{x}_t + 2h\omega \dot{x}_t + \omega^2 x_t = -\ddot{y}_t \quad (2.48)$$

$$\ddot{x}_{t+\Delta t} + 2h\omega \dot{x}_{t+\Delta t} + \omega^2 x_{t+\Delta t} = -\ddot{y}_{t+\Delta t} \quad (2.49)$$

Taylor Expansion

$$x_{t+\Delta t} = x_t + \dot{x}_t \cdot \Delta t + \ddot{x}_{t_2} \cdot \frac{(\Delta t)^2}{2} \quad (2.50)$$

$$\dot{x}_{t+\Delta t} = \dot{x}_t + \ddot{x}_{t_1} \cdot \Delta t \quad (2.51)$$

Taylor Expansion

$$f(t + \Delta t) = f(t) + \Delta t \cdot f'(t_1) \quad t \leq t_2 \leq t + \Delta t \quad (2.52)$$

$$f(t + \Delta t) = f(t) + \Delta t \cdot f'(t) + f''(t_2) \frac{(\Delta t)^2}{2} \quad t \leq t_2 \leq t + \Delta t \quad (2.53)$$

□) 代数方程式を計算

(1) Newmark の β 法 (内挿方式)

$$\ddot{x}_{t_2} = (1-\gamma)\ddot{x}_t + \gamma \cdot \ddot{x}_{t+\Delta t}, \quad \ddot{x}_{t_1} = (1-2\beta)\ddot{x}_t + 2\beta \ddot{x}_{t+\Delta t} \quad (2.54)$$

$$x_{t+\Delta t} = x_t + \dot{x}_t \Delta t + [(1-2\beta)\ddot{x}_t + 2\beta \ddot{x}_{t+\Delta t}] \frac{(\Delta t)^2}{2} \quad (2.55)$$

$$\dot{x}_{t+\Delta t} = \dot{x}_t + [(1-\gamma)\ddot{x}_t + \gamma \ddot{x}_{t+\Delta t}] \Delta t \quad (2.56)$$

$$[A] = [?]$$

$$[B] = [?]$$

(2) Wilson の θ 法 (外挿方式)

時間 $t \sim t + \Delta t$ で成立する運動方程式が、時間 $t + \theta\Delta t$ ($\theta \geq 1$) でも成立するとする。

$$\ddot{x}_{t+\theta\Delta t} + 2h\omega \dot{x}_{t+\theta\Delta t} + \omega^2 x_{t+\theta\Delta t} = -\ddot{y}_{t+\theta\Delta t}, \quad \theta \geq 1 \quad (2.57)$$

$$\left. \begin{aligned} \ddot{x}_{t+\Delta t} &= (1-\theta)\ddot{x}_t + \theta \cdot \ddot{x}_{t+\Delta t} \\ \ddot{y}_{t+\Delta t} &= (1-\theta)\ddot{y}_t + \theta \cdot \ddot{y}_{t+\Delta t} \end{aligned} \right\} \quad (2.58)$$

$$\left. \begin{aligned} x_{t+\theta\Delta t} &= x_t + \dot{x}_t(\theta\Delta t) + \left[\left(1 - \frac{\theta}{3}\right)\ddot{x} + \frac{\theta}{3}\ddot{x}_{t+\theta\Delta t} \right] \frac{(\theta\Delta t)^2}{2} \\ \dot{x}_{t+\theta\Delta t} &= \dot{x}_t + \left[\left(1 - \frac{\theta}{2}\right)\ddot{x} + \frac{\theta}{2}\ddot{x}_{t+\theta\Delta t} \right] (\theta\Delta t) \end{aligned} \right\} \quad (2.59)$$

In the case of $\theta=1$,

$$\left. \begin{aligned} x_{t+\Delta t} &= x_t + \dot{x}_t(\Delta t) + \left[\frac{2}{3}\ddot{x} + \frac{1}{3}\ddot{x}_{t+\Delta t} \right] \frac{(\Delta t)^2}{2} \\ \dot{x}_{t+\Delta t} &= \dot{x}_t + \left[\frac{1}{2}\ddot{x} + \frac{1}{2}\ddot{x}_{t+\Delta t} \right] (\Delta t) \end{aligned} \right\} \quad (2.60)$$

= **Linear Acceleration Method** ($\gamma=0.5, \beta=1/6$)

- 地震応答スペクトル (*Earthquake Response Spectrum*)

(相対) 変位応答スペクトル(*Displacement Response Spectrum*): x

(相対) 速度応答スペクトル(*Velocity Response Spectrum*): \dot{x}

(絶対) 加速度応答スペクトル(*Acceleration Response Spectrum*): $\ddot{x} + \ddot{y}$

$S_d(h, T)$: 最大相対変位応答値(*Displacement Response Spectrum*)

$S_v(h, T)$: 最大相対速度応答値(*Velocity Response Spectrum*)

$S_a(h, T)$: 最大絶対加速度応答値(*Acceleration Response Spectrum*)

Plots of maximum response values against selected parameters of the system or of forcing function (earthquake considered) are called 'response spectra'.

For one-degree system, the natural period (or frequency) is the characteristic that determined its response to a given forcing function

□ ratio of maximum dynamic stress in a structure to the corresponding static stress

Calculation results of response for each of systems

one-degree systems with different natural periods (frequency)

input force (earthquake)

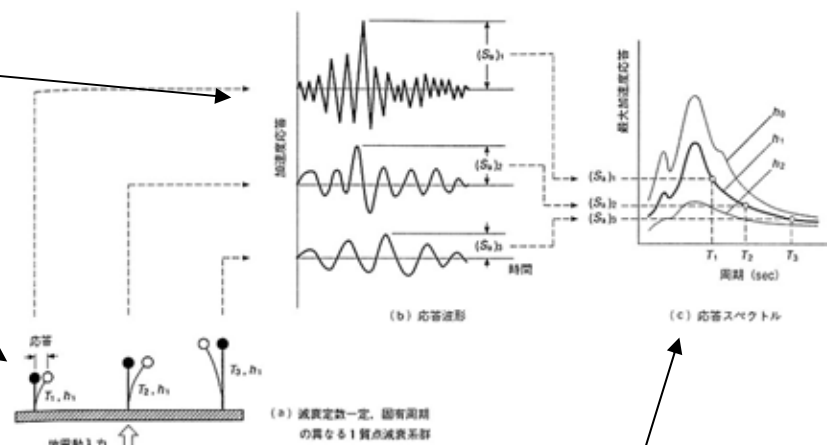


図7-11 地震応答スペクトルの説明図

Maximum response values of systems

$$S_d = \frac{1}{\omega} S_v = \frac{T}{2\pi} S_v : \text{最大相対変位応答値(Displacement Response Spectrum)}$$

$$S_v = S_v : \text{最大相対速度応答値(Velocity Response Spectrum)}$$

$$S_a = \omega S_v = \frac{2\pi}{T} S_v : \text{最大絶対加速度応答値(Acceleration Response Spectrum)}$$

(2.61)

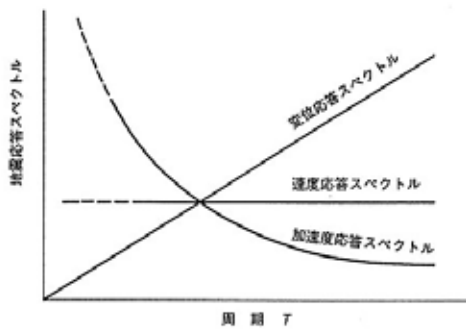


図7-16 地震応答スペクトルの概略の形

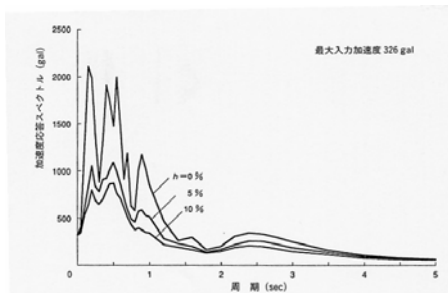


図7-12 エル・セントロ地震波の加速度応答スペクトル

S_a vs period

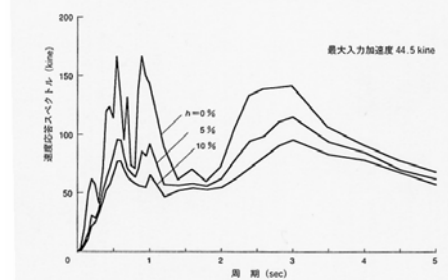


図7-13 エル・セントロ地震波の速度応答スペクトル

S_v vs period

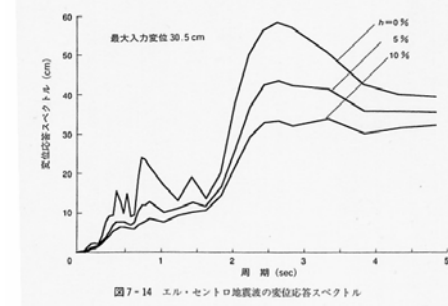


図7-14 エル・セントロ地震波の変位応答スペクトル

S_D vs period

- 3重応答スペクトル (Tripartite response spectrum)

$$\left. \begin{aligned} \log S_v &= \log S_a - \log(2\pi) + \log T \\ \log S_v &= \log S_d + \log(2\pi) - \log T \end{aligned} \right\} \quad (2.62)$$

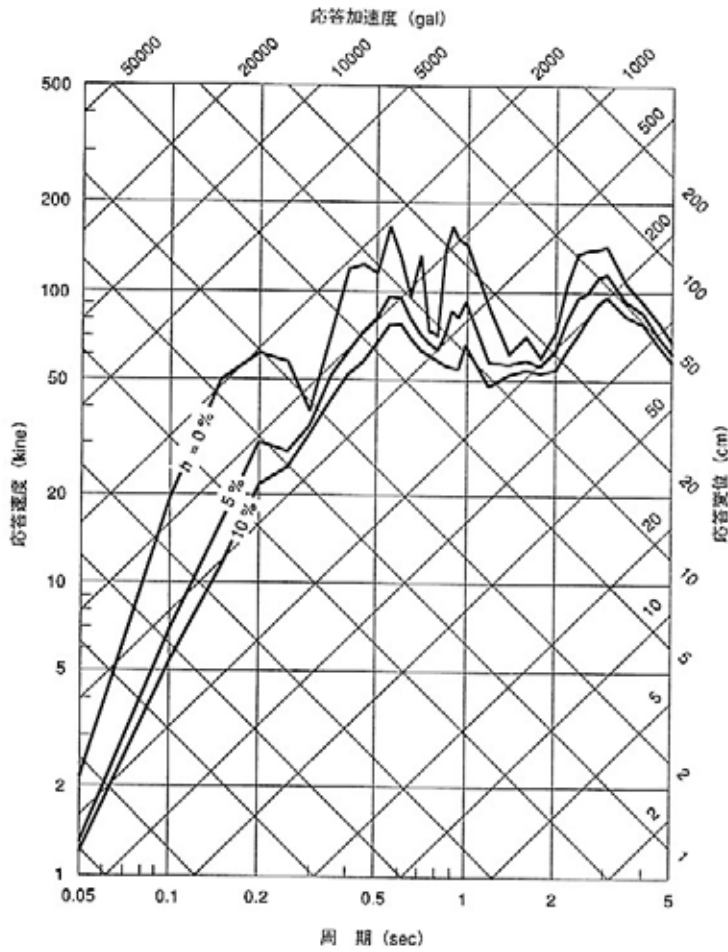


図7-19 エル・セントロ地震波の3重応答スペクトル

- 応答スペクトル (*response spectrum*) について

- ・フーリエ・スペクトルは地震波そのものの周波数特性。応答スペクトルは構造物 (1 質点減衰系) を含んだ地震動の全体像
- ・モード解析 (modal analysis)

加速度応答スペクトル: 地震力、ベース・シア係数(base shear coefficient)、動的震度

地震力

$$Q_{\max} = m(\ddot{x} + \ddot{y})_{\max} \quad (2.63)$$

ベース・シア係数(base shear coefficient)、動的震度

$$C = \frac{Q_{\max}}{W} = \frac{(\ddot{x} + \ddot{y})_{\max}}{g} = \frac{S_a(h, T)}{g} \quad (2.64)$$

静的震度: 静的耐震設計 k_h

$$Q_{\max} = k_h \cdot W \quad (2.65)$$

速度応答スペクトル: 地震動が構造物に与える最大のエネルギー

$$\text{最大ひずみエネルギー (strain energy): } \frac{1}{2} k (x_{\max})^2 \quad (2.66)$$

$$\text{単位質量あたりの最大エネルギー: } \frac{1}{2} \cdot \frac{k}{m} (x_{\max})^2 = \frac{1}{2} (\omega x_{\max})^2 = \frac{1}{2} S_v^2 \quad (2.67)$$

(maximum energy per unit mass)

$$\text{スペクトル強度 (Spectral intensity): } I_h = \int_{0.1}^{2.5} S_v(h, T) dT \quad (2.68)$$

変位応答スペクトル: ひずみの大きさ ~ 応力の大きさ

$$\text{最大せん断力: } Q_{\max} = k \cdot x_{\max} \quad (2.69)$$

$$m \cdot (\ddot{x} + \ddot{y})_{\max} = k x_{\max} \quad (2.70)$$

[Recommended text]

William Weaver, Jr., Stephen P. Timoshenko, Donovan H. Young; Vibration problems in engineering, 5th edition(?), Wiley Interscience.