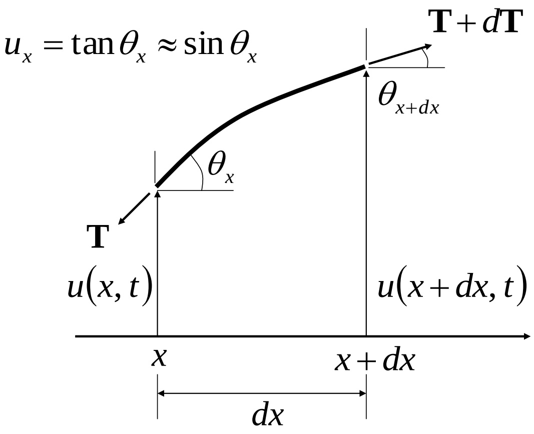
3. 微分方程式の導出(Introduction of PDE of 2nd order)

3.1 一次元波動方程式の導入(Introduction to Wave Propagation in 1D)



振動する弦の小片（x，$x+\Delta x$）と作用する力
Infinitesimal element （x，$x+\Delta x$） of bowstring in 1D vibration and forces T (internal tension) applied to it.

- 微分表記（Detonations for differential）

$$u = u(x,t),\ u_x = \frac{\partial u}{\partial x},\ u_t = \frac{\partial u}{\partial t},\ u_{xx} = \frac{\partial^2 u}{\partial x^2},\ u_{tt} = \frac{\partial^2 u}{\partial t^2}$$

- 微小振幅（Infinitesimal Amplitude）

$$\theta \to 0 \Rightarrow \cos\theta \to 1,\ \sin\theta \to \theta,\ \tan\theta \to \theta:\ \theta_x \approx \left.\frac{\partial u}{\partial x}\right|_x$$

- 釣合い式・運動方程式（Equilibrium / Motion Equation）

$$-T\cos\theta_x + (T+dT)\cos\theta_{x+dx} = 0 \qquad \textit{x-direction}$$
$$-\rho\cdot dx\cdot u_{tt} - T\sin\theta_x + (T+dT)\sin\theta_{x+dx} = 0 \qquad \textit{y-direction}$$

$$\downarrow$$

$$dT \approx 0 \qquad \textit{x-direction}$$
$$-\rho\cdot dx\cdot u_{tt} - T(u_x - u_{x+dx}) = 0 \qquad \textit{y-direction}$$

$$\downarrow$$

$$T(x,t) = T(x+dx,t) \qquad \textit{x-direction}$$

$$u_{tt} = \frac{T}{\rho}u_{xx} = \left(\frac{T}{\rho}\frac{u_{x+dx} - u_x}{dx}\right)_{dx\to 0} \qquad \textit{y-direction}$$

4. 微分方程式の解析解（Analytical solution of PDE）

4.1 波動方程式の D'Alembert 解（D'Alembert's Solution for Differential Equations for Wave Propagation）

Differential Equation

$$u_{tt} = V^2 u_{xx}, (-\infty < x < \infty,\ 0 < t < \infty) \tag{1}$$

Initial Conditions: We must two initial conditions to solve wave propagation with second differential equation

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}, (-\infty < x < \infty) \tag{2}$$

D'Alembert's Solution

1.- Replacement: $\{x, t\} \Rightarrow \{\xi, \eta\}$

$$\begin{cases} \xi = x + Vt \\ \eta = x - Vt \end{cases} \tag{3}$$

$$u_{\xi\eta} = 0 \tag{4}$$

$$\begin{cases} u_x = u_\xi + u_\eta \\ u_t = V(u_\xi - u_\eta) \\ u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \\ u_{tt} = V^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) \end{cases} \tag{5}$$

2.- Integration

$$u_\eta(\xi,\eta) = \phi(\eta);\ \text{Arbitrary Function}\ \ \phi(\eta)\ \text{ as }\ \eta \tag{6}$$

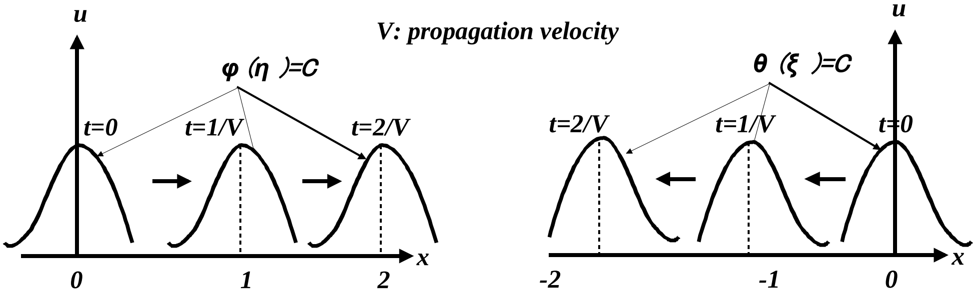
$$u(\xi,\eta) = \Phi(\eta) + \theta(\xi) \tag{7}$$

Arbitrary Function $\theta(\xi)$ as ξ

$$\Phi(\eta) = \int \phi(\eta)\, d\eta$$

3.- Replacement: $\{\xi, \eta\} \Rightarrow \{x, t\}$

$$u(x,t) = \phi(x - Vt) + \theta(x + Vt) \tag{8}$$



4.- Initial Condition:

$$u(x,t) = \phi(x - Vt) + \theta(x + Vt)$$

$$\begin{cases} u(x,0) = f(x) \\ u_t(x,0) = g(x) \end{cases}$$

$$\begin{cases} \phi(x) + \theta(x) = f(x) \\ -V\phi'(x) + V\theta'(x) = g(x) \end{cases} \tag{9}$$

$$-V\phi(x) + V\theta(x) = \int_{x_0}^{x} g(\xi)d\xi + K \tag{10}$$

$$\phi(x) = \frac{1}{2} f(x) - \frac{1}{2V}\int_{x_0}^{x} g(\xi)d\xi - K \tag{11}$$

$$\theta(x) = \frac{1}{2} f(x) + \frac{1}{2V}\int_{x_0}^{x} g(\xi)d\xi + K \tag{12}$$

$$u(x,t) = \frac{1}{2}\left[f(x - Vt) + f(x + Vt)\right] + \frac{1}{2V}\int_{x-Vt}^{x+Vt} g(\xi)d\xi \tag{13}$$

– 特性曲線(Characteristic Curve)

Example-1

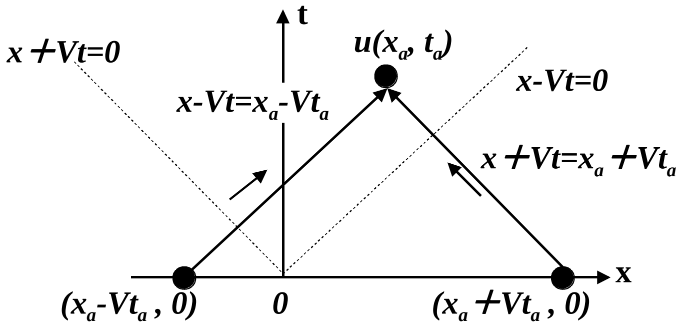
Initial condition

$$\begin{cases} u(x,0)=f(x) \\ u_t(x,0)=g(x)=0 \end{cases} \tag{14}$$

$$u(x,t)=\frac{1}{2}\left[f(x-Vt)+f(x+Vt)\right] \tag{15}$$

characteristic curve

$$\begin{cases} x-Vt=x_a-Vt_a \\ x+Vt=x_a+Vt_a \end{cases} \tag{16}$$



Example-2

$$\begin{cases} u(x,0)=f(x)=\begin{cases} 1 & (-1\le x\le 1) \\ 0 & \text{the other points} \end{cases} \\ \qquad u_t(x,0)=0 \end{cases} \tag{17}$$

Characteristic curve

x = -1 or 1 , t=0

$$\begin{cases} x-Vt=\pm 1 \\ x+Vt=\pm 1 \end{cases} \tag{18}$$