フーリエ変換と地震のスペクトル分析

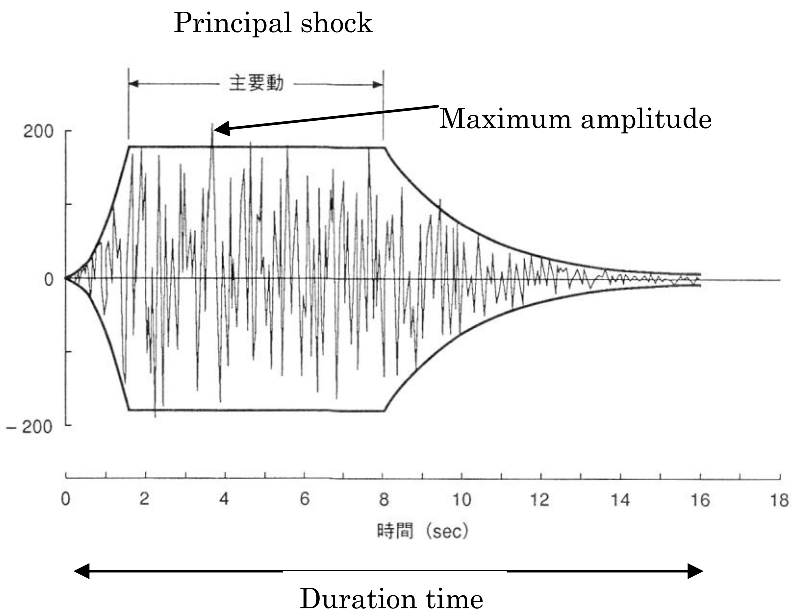
Spectrum Analysis of Earthquake and Fourier Transform

* このプリントは「新・地震動のスペクトル解析入門」（大崎順彦，鹿島出版会）を元に作成しています。：The original of this document was from the above text book.

- **世界最初の強震記録**： **強震計**（Strong-Motion Acceleration: SMAC）
  - ・Imperial Valley Earthquake at EL Centro(1940.5.18)　326gal：人類初の強震の記録
    (Record of the first strong motion due to earthquake in the human history)
  - ・Arvin-Tahachapi Earthquake at Taft(1952.7.12)　147gal

- **地震波の特長**（Properties of Earthquake Wave）

  対象とする地盤－構造物系に対して与える影響を考えるための有力な手がかり。
  - ・最大振幅（maximum amplitude）
  - ・継続時間（duration time）
  - ・包絡曲線（envelope curve）：主要動（principal shock）
  - ・波数（numbers of wave）
  - ・振動周期（periods）
  - ・エネルギ（Energy）



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Finite Fourier Approximation of Time History and Time Series and its Formulations

1) Approximation of digital time history data with Tri-angle series

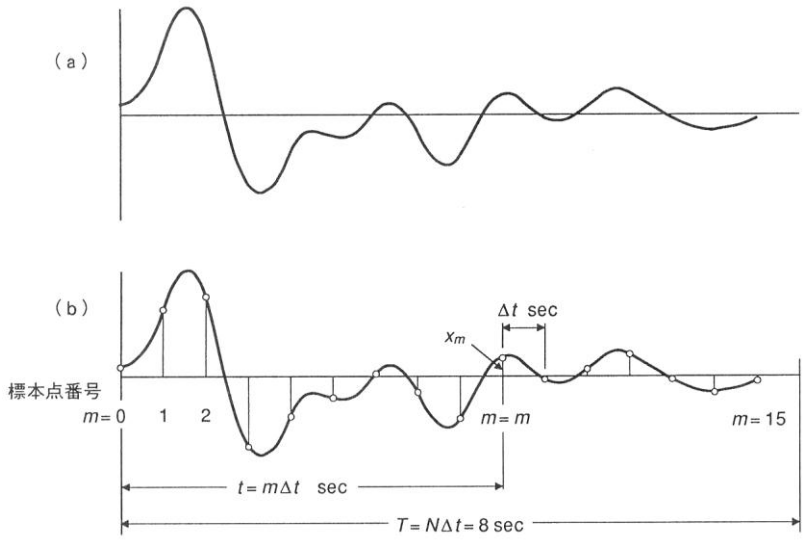
Discrete System: $\Delta t > 0$: data sampling interval

$(t_0, x_0),\ (t_1, x_1),\ (t_2, x_2), \ldots,\ (t_{N-1}, x_{N-1})$ ・・…N =(N-1) +1: data: *N conditions*

Duration Time: $T = N\ \Delta t$ (1.1)

Time: $t = m\ \Delta t$ (m=0, 1, 2, ・..., N-1) (1.2)

A data point: $x_m = x(m\ \Delta t)$



N=16(m=0 – 15) : x_0, x_1, …….., x_{15}.
duration time: T=16×Δt

2) Approximation of digital time history data with <u>infinite</u> tri-angle series

$$\left.\begin{aligned} &A_0 + A_1 \cos t + A_2 \cos 2t + \cdots + A_k \cos kt + \cdots \\ &+ B_0 + B_1 \sin t + B_2 \sin 2t + \cdots + B_k \sin kt + \cdots \end{aligned}\right\} \quad (2.1)$$

What is the period T_p of cos(kt)or sin(kt)?

$$\cos kt = \cos k(t + T_p) = \cos(kt + 2\pi) = \cos k\left(t + \frac{2\pi}{k}\right)$$

$\therefore \quad T_p = \dfrac{2\pi}{k}$: As k increases, the period T_p decrease (the frequency f=1/T_p increases).

$$\sum_{k=0}^{\infty}\left[A_k \cos kt + B_k \sin kt\right] \tag{2.2}$$

replace t by $\frac{2\pi}{T}t$ or $\frac{2\pi}{N\Delta t}t$

$$\sum_{k=0}^{\infty}\left[A_k \cos\frac{2\pi kt}{N\Delta t} + B_k \sin\frac{2\pi kt}{N\Delta t}\right] \tag{2.3}$$

3) Approximation of digital time history data with <u>finite</u> triangle series

<u>Set *k* to be from 0 to *N*/2</u>

$$x_m = \sum_{k=0}^{N/2}\left[A_k \cos\frac{2\pi kt}{N\Delta t} + B_k \sin\frac{2\pi kt}{N\Delta t}\right] = \sum_{k=0}^{N/2}\left[A_k \cos\frac{2\pi km}{N} + B_k \sin\frac{2\pi km}{N}\right] \tag{3.1}$$

$$\left.\begin{matrix} A_0, A_1, A_2, \cdots, A_k, \cdots, A_{N/2} \\ B_0, B_1, B_2, \cdots, B_k, \cdots, B_{N/2} \end{matrix}\right\} \quad \text{Here, Number of unknown coefficients is } 2(N/2+1) \tag{3.2}$$

Number of unknown coefficient 2(N/2+1) =N+2 > number of conditions (data) N

<u>From partial consideration,</u>

<u>For the case of</u> $k=0$, $A_0 \cos\frac{2\pi km}{N} = A_0 \cdot 1 = A_0$ and $B_0 \sin\frac{2\pi km}{N} = B_0 \cdot 0 \equiv 0$ (3.3)

<u>For the case of</u> $k = N/2$, $B_{N/2} \sin\frac{2\pi km}{N} = B_{N/2} \sin \pi m \equiv 0$ (3.4)

<u>Consequently, Eq.(3.1) is reduced to</u>

$$x_m = A_0 + \sum_{k=1}^{N/2-1}\left[A_k \cos\frac{2\pi km}{N} + B_k \sin\frac{2\pi km}{N}\right] + A_{N/2} \cos\frac{2\pi(N/2)m}{N} \tag{3.5}$$

For convenience

$$x_m = A_0/2 + \sum_{k=1}^{N/2-1}\left[A_k \cos\frac{2\pi km}{N} + B_k \sin\frac{2\pi km}{N}\right] + A_{N/2}/2 \cos\frac{2\pi(N/2)m}{N} \tag{3.6}$$

$$\left.\begin{matrix} A_0, A_1, A_2, \cdots, A_k, \cdots, A_{N/2-1}, A_{N/2} \\ B_1, B_2, \cdots, B_k, \cdots, B_{N/2-1} \end{matrix}\right\} \quad N/2+1+N/2-1=N \tag{3.7}$$

<u>Therefore,</u>

Numbers of unknown coefficient N = Condition Equation N (3.8)

4) Determination of A_k and B_k with orthogonal property of triangle functions

<u>なぜ三角関数か？/ Why do we use triangle functions for Fourier approximation?</u>

<u>三角関数系の直交性を利用する**(We utilize o*rthogonal property for Triangle Functions)***</u>

$$2\cos\alpha\cdot\cos\beta=\cos(\alpha+\beta)+\cos(\alpha-\beta) \quad \text{(a)}$$

$$2\cos\alpha\cdot\sin\beta=\cos(\alpha+\beta)-\sin(\alpha-\beta) \quad \text{(b)}$$

$$2\sin\alpha\cdot\sin\beta=-\cos(\alpha+\beta)+\cos(\alpha-\beta) \quad \text{(c)}$$

$$2\cos^2\alpha=1+\cos 2\alpha \quad \text{(d)}$$

$$2\sin^2\alpha=1-\cos 2\alpha \quad \text{(e)}$$

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots+\cos\{\alpha+(N-1)\beta\} =\frac{\cos\left(\alpha+\frac{N-1}{2}\beta\right)\sin\frac{N\beta}{2}}{\sin\frac{\beta}{2}} \quad \text{(f)}$$

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots+\sin\{\alpha+(N-1)\beta\} =\frac{\sin\left(\alpha+\frac{N-1}{2}\beta\right)\sin\frac{N\beta}{2}}{\sin\frac{\beta}{2}} \quad \text{(g)}$$

if $\alpha=0$, summarize the results.

$$\sum_{m=0}^{N-1}\cos\beta m=\frac{\cos\frac{N-1}{2}\beta\cdot\sin\frac{N\beta}{2}}{\sin\frac{\beta}{2}} \quad \text{(h)}$$

$$\sum_{m=0}^{N-1}\sin\beta m=\frac{\sin\frac{N-1}{2}\beta\cdot\sin\frac{N\beta}{2}}{\sin\frac{\beta}{2}} \quad \text{(i)}$$

$$\left.\begin{array}{l}
\displaystyle\sum_{m=0}^{N-1}\cos\frac{2\pi lm}{N}\cos\frac{2\pi km}{N}=\begin{cases}N/2 & k=l\\ 0 & k\neq l\end{cases}\\
\displaystyle\sum_{m=0}^{N-1}\sin\frac{2\pi lm}{N}\sin\frac{2\pi km}{N}=\begin{cases}N/2 & k=l\\ 0 & k\neq l\end{cases}\\
\displaystyle\sum_{m=0}^{N-1}\sin\frac{2\pi lm}{N}\cos\frac{2\pi km}{N}=0
\end{array}\right\} \quad \text{(j)}$$

For A_k

たとえば、A_k を求める。/ For example, we calculate the factor A_k,

$$x_m = A_0/2 + \sum_{l=1}^{N/2-1}\left[A_l \cos\frac{2\pi lm}{N} + B_l \sin\frac{2\pi lm}{N}\right] + A_{N/2}/2\cos\frac{2\pi(N/2)m}{N} \qquad (4.1)$$

1)上式の両辺に $\cos(2\pi km/N)$ を掛ける / Multiplication of $\cos(2\pi km/N)$ to Eq.(4.1).

$$\begin{aligned} x_m \cos(2\pi km/N) = {} & \frac{A_0}{2}\cos\frac{2\pi km}{N} \\ & + \sum_{l=1}^{N/2-1}\left[A_l \cos\frac{2\pi lm}{N}\cos\frac{2\pi km}{N} + B_l \sin\frac{2\pi lm}{N}\cos\frac{2\pi km}{N}\right] \\ & + \frac{A_{N/2}}{2}\cos\frac{2\pi(N/2)m}{N}\cos\frac{2\pi km}{N} \end{aligned} \qquad (4.2)$$

2) $m=0$ から $m=N-1$ まで総和をとる / Summation from m=0 to m=N-1 in Eq. (4.2)

$$\begin{aligned} \sum_{m=0}^{N-1} x_m \cos(2\pi km/N) = {} & \frac{A_0}{2}\sum_{m=0}^{N-1}\cos\frac{2\pi km}{N} \quad (\text{ -> 0 }) \\ & + \sum_{l=1}^{N/2-1}\left[\sum_{m=0}^{N-1} A_l \cos\frac{2\pi lm}{N}\cos\frac{2\pi km}{N}\right] \\ & + \sum_{l=1}^{N/2-1}\left[\sum_{m=0}^{N-1} B_l \sin\frac{2\pi lm}{N}\cos\frac{2\pi km}{N}\right] (\text{ -> 0 }) \\ & + \frac{A_{N/2}}{2}\cos\frac{2\pi(N/2)m}{N}\cos\frac{2\pi km}{N} (\text{ -> 0 }) \end{aligned} \qquad (4.3)$$

1st term, 3rd term and 4th term in right formula =0 with account for the orthogonal

$$\sum_{m=0}^{N-1} x_m \cos(2\pi km/N) = \sum_{l=1}^{N/2-1} A_l\left[\sum_{m=0}^{N-1}\cos\frac{2\pi lm}{N}\cos\frac{2\pi km}{N}\right] \qquad (4.4)$$

$$\sum_{l=1}^{N/2-1} A_l\left[\sum_{m=0}^{N-1}\cos\frac{2\pi lm}{N}\cos\frac{2\pi km}{N}\right] = A_1\cdot 0 + A_2\cdot 0 + \cdots + A_k\cdot\frac{N}{2} + \cdots + A_{N/2-1}\cdot 0 \qquad (4.5)$$

$$A_k = \frac{2}{N}\sum_{m=1}^{N-1} x_m \cos\frac{2\pi km}{N} \qquad (4.6)$$

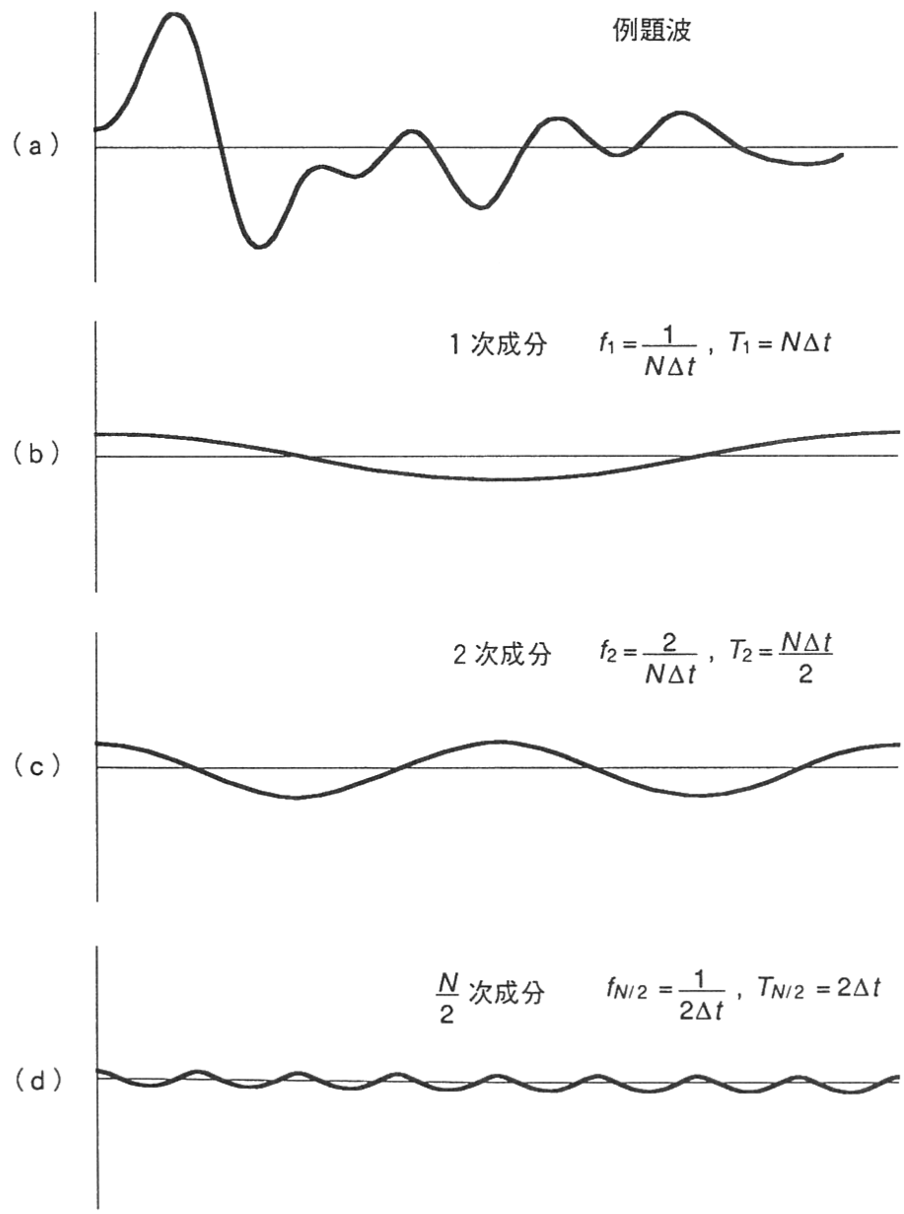
$$\left.\begin{aligned} A_k &= \frac{2}{N}\sum_{m=1}^{N-1} x_m \cos\frac{2\pi km}{N} \qquad k = 0, 1, 2, \cdots, N/2-1, N/2 \\ B_k &= \frac{2}{N}\sum_{m=1}^{N-1} x_m \sin\frac{2\pi km}{N} \qquad k = 1, 2, \cdots, N/2-1 \end{aligned}\right\} \qquad (4.7)$$

$$\frac{A_0}{2} = \frac{1}{N}\sum_{m=0}^{N-1} x_m \text{ : mean value} \qquad (4.8)$$

3)時間関数 $x(t)$ の近似:　*Fourier Approximation*

$$t = m\Delta t\,,\quad m = \frac{t}{\Delta t} \tag{4.9}$$

$$x(t) \approx A_0/2 + \sum_{k=1}^{N/2-1}\left[A_k \cos\frac{2\pi k t}{N\,\Delta t} + B_k \sin\frac{2\pi k t}{N\,\Delta t}\right] + A_{N/2}/2\cos\frac{2\pi (N/2) t}{N\,\Delta t} \tag{4.10}$$



5) Spectrum Properties

$$x(t) \approx A_0/2 + \sum_{k=1}^{N/2-1}\left[A_k \cos\frac{2\pi kt}{N\,\Delta t} + B_k \sin\frac{2\pi kt}{N\,\Delta t}\right] + A_{N/2}/2\cos\frac{2\pi(N/2)t}{N\,\Delta t} \tag{5.1}$$

1)周波数・周期（Frequency/Period）

$$\omega_k = \frac{2\pi}{N\Delta t}k = 2\pi\frac{k}{T} \tag{5.2}$$

$$T_k = \frac{2\pi}{\omega_k} = \frac{T}{k} = \frac{N\Delta T}{k},\quad f_k = \frac{1}{T_k} = \frac{k}{T} = \frac{k}{N\Delta T} \tag{5.3}$$

－ $k = 0$，$f_k = f_0 = 0$: 直流成分(Cascade Component)

$$\frac{A_0}{2} = \frac{1}{N}\sum_{m=0}^{N-1} x_m \quad \text{全体のゼロ点からのズレ} \tag{5.4}$$

－ $k \neq 0$，$f_k \neq 0$:

$$f_1 < f_2 < \cdots f_{N/2-1} < f_{N/2}, \qquad T_1 > T_2 > \cdots T_{N/2-1} > T_{N/2} \tag{5.5}$$

－ 分解する周波数はトビトビ（Discontinuity of Decomposed Frequency）

$$\Delta f \equiv f_{k+1} - f_k = \frac{1}{N\Delta t} \tag{5.6}$$

2)基本振動数(Fundamental Frequency)

$$f_1 = \frac{1}{T_1} = \frac{1}{T} = \frac{1}{N\Delta T} \tag{5.7}$$

3)ナイキスト振動数(Nyquist Frequency)　　分解能：Resolving power

検出可能な高周波数の限界値 /: Limit value of detection possible high frequency

$$f_{N/2} = \frac{1}{T_{N/2}} = \frac{1}{2\Delta T} \tag{5.8}$$

$$\Delta t = 0.01\,(\text{sec.}) \rightarrow f_{N/2} = \frac{1}{2\cdot 0.01} = 50Hz \tag{5.9}$$

4)振幅・位相角(Amplitude/Phase Angle)　　　　　　　　　　情報は 2 つ

$$A_k \cos(\omega t) + \mathbf{B}_k \sin(\omega t) = X_k \cos(\omega t + \phi) \tag{5.10}$$

$$X_k = \sqrt{A_k^2 + B_k^2} \tag{5.11}$$

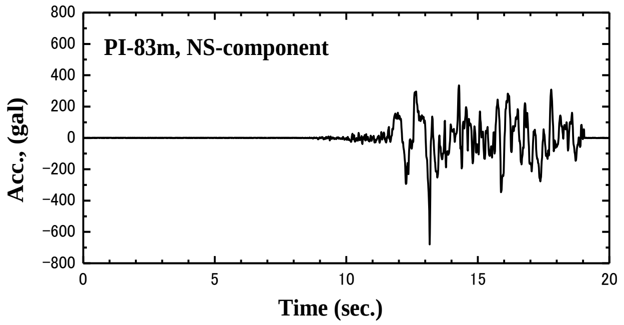
$$\phi = \tan^{-1}\left(-\frac{B_k}{A_k}\right) \tag{5.12}$$

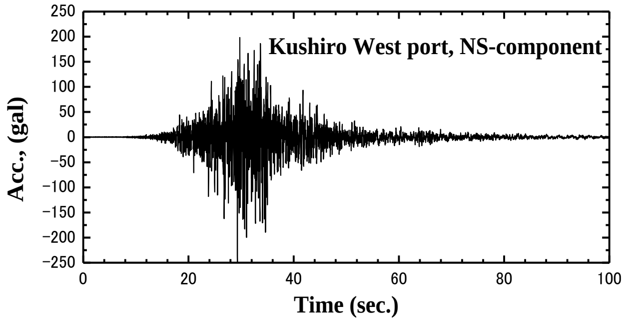
4)フーリエ（振幅）スペクトル(Fourier Amplitude Spectrum/ Fourier Spectrum)

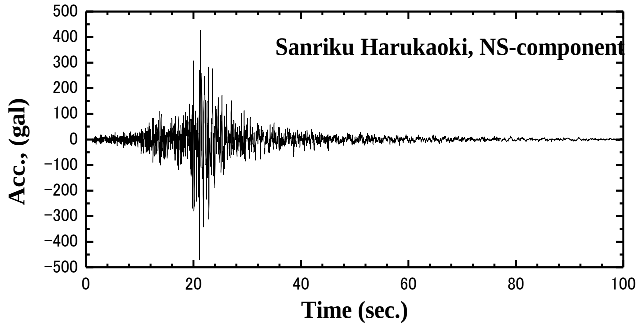
$$\frac{T}{2}X_k \qquad \text{dimension: } [X_k][\text{sec.}] \tag{5.13}$$

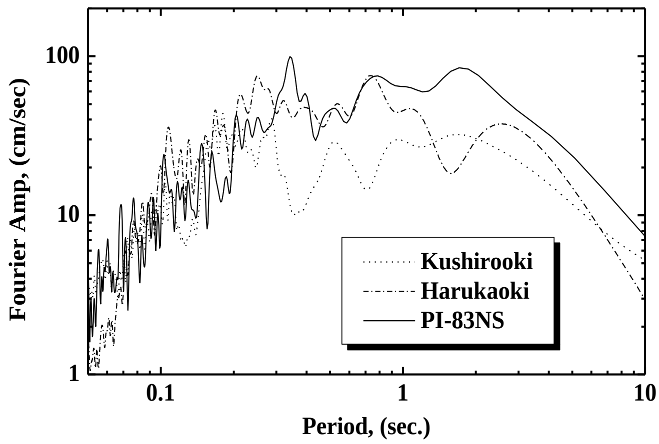
5)パワースペクトル(Power Spectrum): Invariant Value

$$\sum_{m=0}^{N-1} x_m^2 \Delta t = T|C_0|^2 + 2\sum_{k=1}^{N/2-1} T|C_k|^2 + T|C_{N/2}|^2 \qquad C_k\text{: 複素数フーリエ振幅} \tag{5.14}$$


PI-83m, NS-component
Acc., (gal)
Time (sec.)


Kushiro West port, NS-component
Acc., (gal)
Time (sec.)


Sanriku Harukaoki, NS-component
Acc., (gal)
Time (sec.)


Fourier Amp, (cm/sec)
Period, (sec.)
Kushirooki
Harukaoki
PI-83NS

6) Finite Fourier Approximation with Complex Number

$$c = a + ib: \ c: \text{complex number}, \ a: \text{real part}, \ b: \text{imaginary part}, \ i = \sqrt{-1} \qquad (6.1)$$

$$|c| = \sqrt{a^2 + b^2} \ : \text{absolute value} \qquad (6.2)$$

$$c \cdot c^* = |c|^2, \ c^* = a - ib: \text{conjugate complex number} \qquad (6.3)$$

$$e^{\pm i\theta} = (\cos\theta \pm i \sin\theta): \text{Euler's Formula} \qquad (6.4)$$

$$\left.\begin{aligned} \cos\theta &= \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right) \\ \sin\theta &= \frac{1}{2}\left(e^{i\theta} - e^{-i\theta}\right) \end{aligned}\right\} \qquad (6.5)$$

Approximation with Complex Number

$$\left.\begin{aligned} \cos\frac{2\pi km}{N} &= \quad \frac{1}{2}\left[e^{i(2\pi km/N)} + e^{-i(2\pi km/N)}\right] \\ \sin\frac{2\pi km}{N} &= -i\frac{1}{2}\left[e^{i(2\pi km/N)} + e^{-i(2\pi km/N)}\right] \end{aligned}\right\} \qquad (6.6)$$

Finite series

$$x_m = \sum_{k=0}^{N-1} C_k e^{i\frac{2\pi km}{N}}, \quad m = 0, 1, 2, \ldots\ldots, N\text{-}1 \qquad (6.7)$$

$$C_k = \frac{A_k - iB_k}{2}, \quad k = 0, 1, 2, \ldots\ldots, N\text{-}1 \qquad N \qquad (6.8)$$

Determination of C_k

$$C_k = \frac{1}{N}\sum_{m=0}^{N-1} x_m e^{-i\frac{2\pi km}{N}}, \quad k = 0, 1, 2, \ldots\ldots, N\text{-}1 \qquad N \qquad (6.9)$$

$$C_k = C_{N-k}: \textit{folding frequency} \quad f_{N/2} = \frac{1}{2\Delta t} \qquad (6.10)$$

$$\left.\begin{aligned} A_k &= 2\,\mathrm{Re}al(C_k) \\ B_k &= \ \ 2\,\mathrm{Im}(C_k) \end{aligned}\right\}, \quad k = 0, 1, 2, \ldots\ldots, N/2 \qquad (6.11)$$

7) Fast Fourier Transform（FFT）

$C_0 \quad C_1 \quad C_2 \quad C_3 \quad C_4 \quad C_5 \quad C_6 \quad C_7$ 　　1×<u>8</u>

一回分割

$C_0 \quad C_2 \quad C_4 \quad C_6$ 　　2×<u>4</u>

$C_1 \quad C_3 \quad C_5 \quad C_7$

2 回分割

$C_0 \quad C_4$ 　　4×<u>2</u>

$C_2 \quad C_6$

$C_1 \quad C_5$

$C_3 \quad C_7$

3 回分割

C_0 　　8×<u>1</u>

C_4

C_2

C_6

C_1

C_5

C_3

C_7

ー **計算時間**(Time for Fourier Coefficient Calculations): T_{cal}

Fourier Transform(FT) $T_{cal} \propto N^2$

Fast Fourier Transform(FFT) $T_{cal} \propto N \log_2 N$

Comparison for Cal. Time

N	Factor	Ratio for T_{cal}
4094	2×23×89	12.9
4095	3^2×5×7×13	3.9
4096	2^{12}	1
4097	17×241	28
4098	2×3×683	77
4099	-	460
4100	2^2×5^2×41	6.3

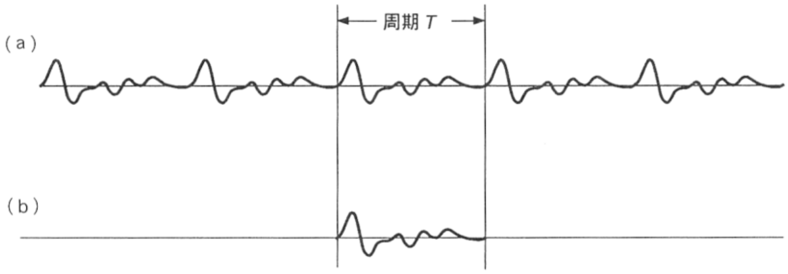
ー **後続のゼロ**(Trailing Zero):

・N が 2 累乗でないとき、ゼロを後続させる。

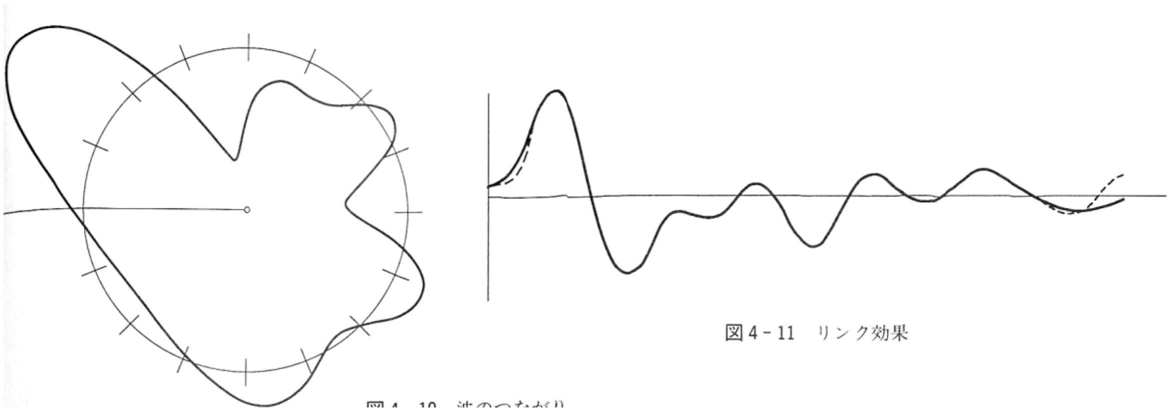
N=3000 → N=3000+1096=4096=2^{12}

・リンク効果(Link Effect)を解消

8) Link effect

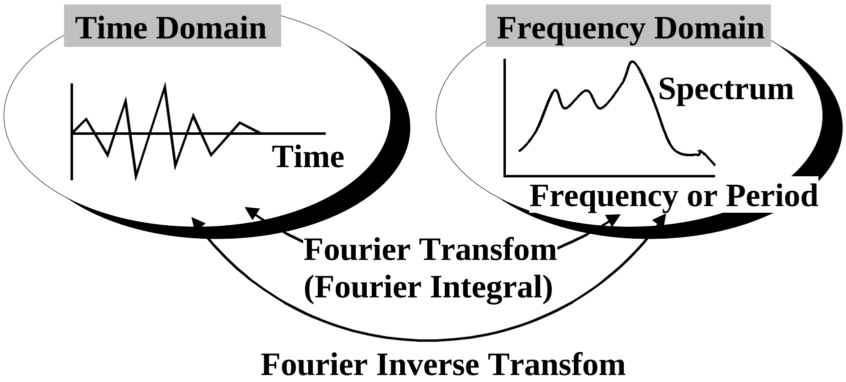


(a) Periodic Function: earthquake motion transformed by Fourier series
(b) Non-periodic function: real earthquake motion



Link Effect in Fourier transform

9) Fourier Integral: Discrete system / Continuous System



$$x(t) = \sum_{k=-\infty}^{\infty} C_k e^{i\frac{2\pi kt}{T}} = \sum_{k=-\infty}^{\infty} (TC_k) e^{i\frac{2\pi kt}{T}} \frac{1}{T} \quad \text{: for } \textit{Discrete System} \tag{9.1}$$

$$C_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-i\frac{2\pi kt}{T}} dt, \quad -\infty \le k \le \infty \tag{9.2}$$

$$f_k = \frac{k}{T}, \ \Delta f = f_{k+1} - f_k = \frac{1}{T} \tag{9.3}$$

$$T \to \infty \quad \begin{cases} \dfrac{k}{T} \to f \\ \Delta f = \dfrac{1}{T} \to df \approx 0 \\ TC_k : discrete \to F(f) : continuos \ \ function \end{cases} \tag{9.4}$$

$$T \to \infty \quad \begin{cases} \dfrac{k}{T} \to f \\ \Delta f = \dfrac{1}{T} \to df \approx 0 \\ TC_k : discrete \to F(f) : continuos \ \ function \\ x(t) = \sum_{k=-\infty}^{\infty} (TC_k) e^{i\frac{2\pi kt}{T}} \dfrac{1}{T} \quad \to \quad x(t) = \int_{-\infty}^{\infty} F(f) e^{i2\pi ft} df \\ TC_k = \dfrac{1}{T}\int_{-T/2}^{T/2} x(t) e^{-i\frac{2\pi kt}{T}} dt \quad \to \quad F(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt \end{cases} \tag{9.5}$$

Fourier transform(Fourier integral): $x(t) \ \to \ F(f) = \lim_{T\to\infty}(TC_k)$ (9.6)

Fourier inverse transform: $F(f) \ \to \ x(t)$ (9.7)

Fourier Spectrum: $T|C_k|$

$$TC_k = \frac{T}{2}(A_k - iB_k) \tag{9.8}$$

$$T|C_k| = \frac{T}{2}\sqrt{A_k^2 + B_k^2} = \frac{T}{2} X_k \tag{9.9}$$

10) Smoothing / Filters

a) Data Window

b) Spectral Windouw

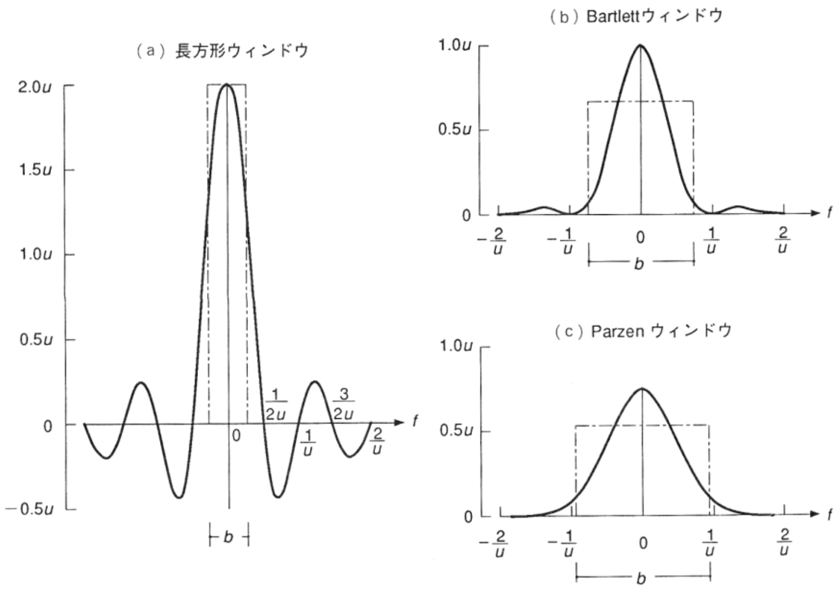
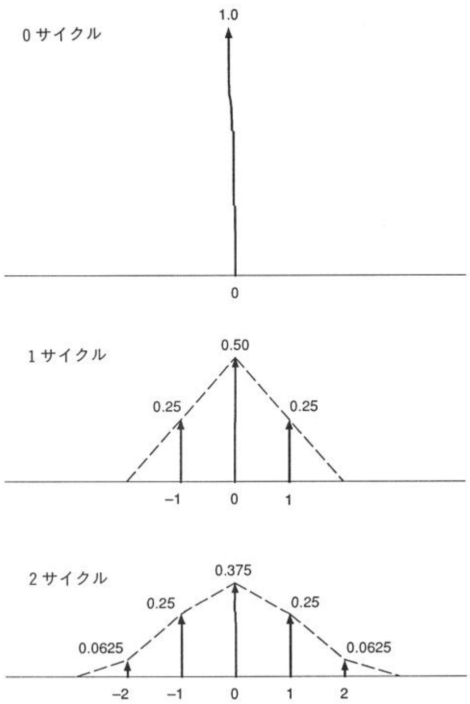
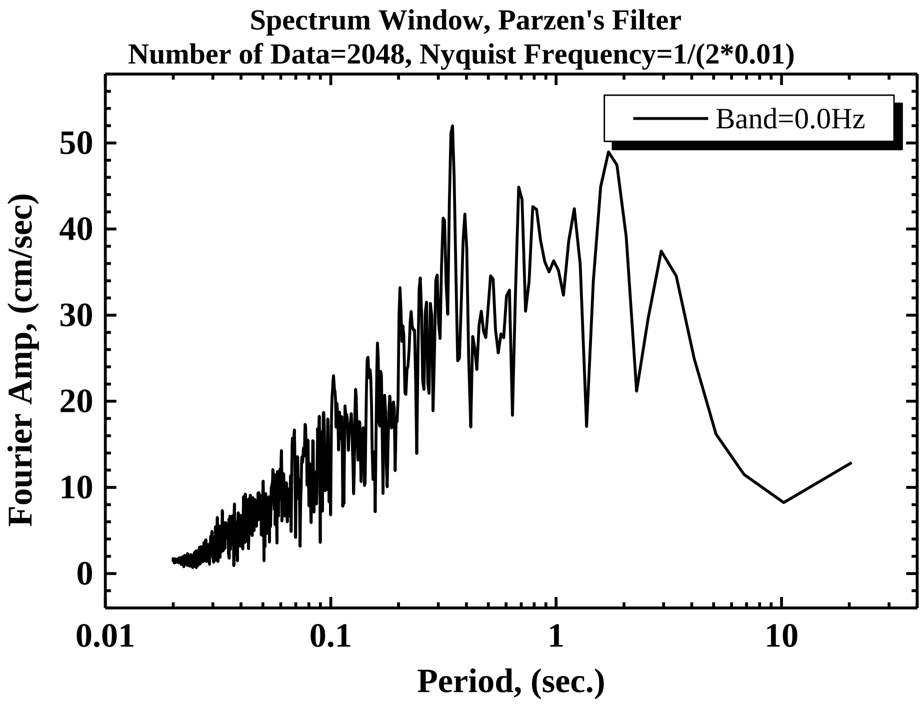
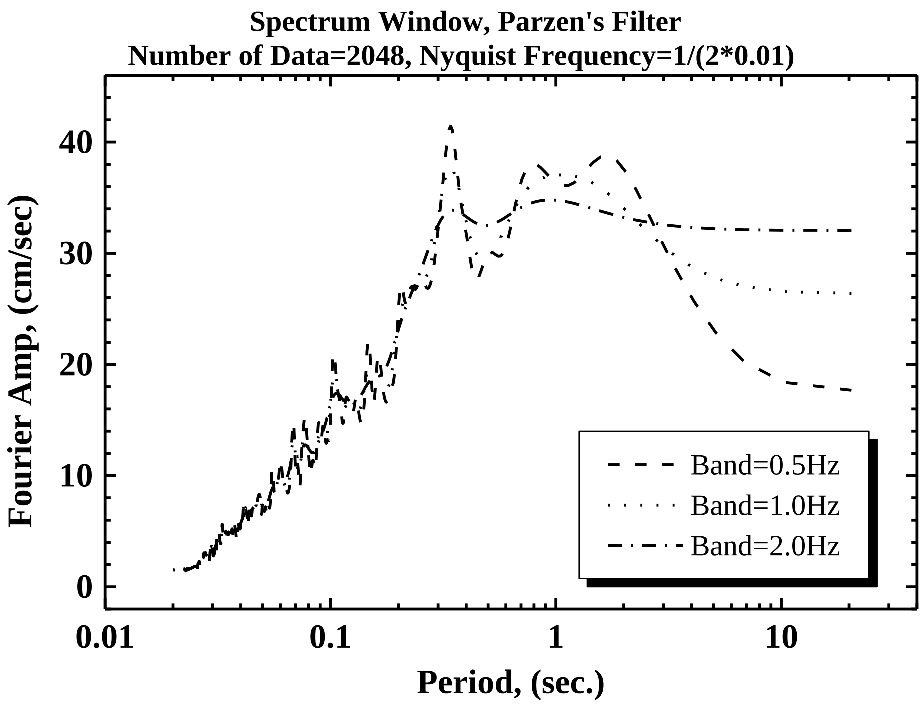


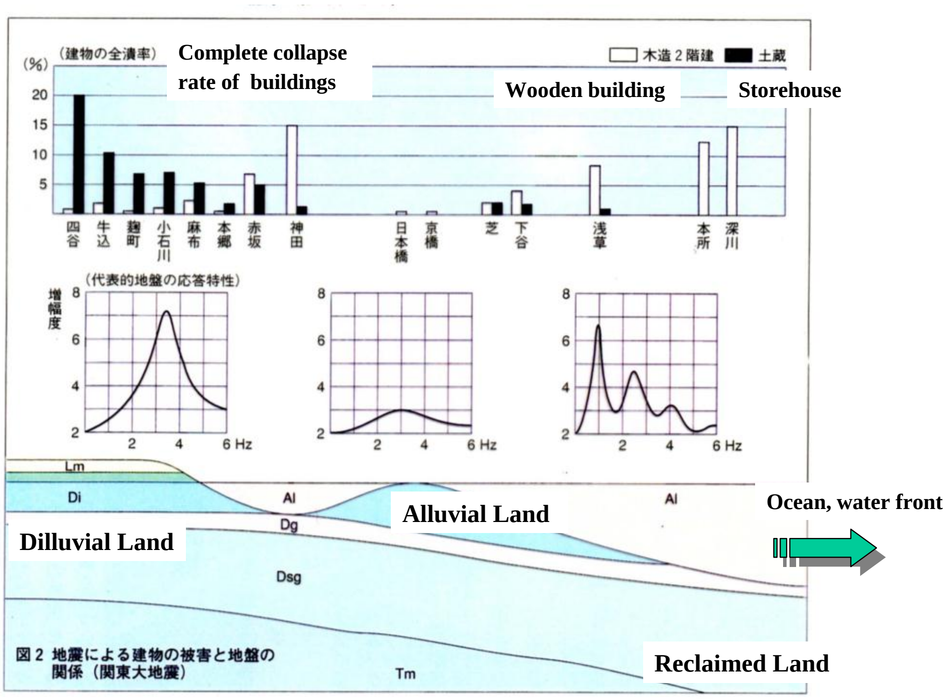
図6－6　スペクトル・ウィンドウ



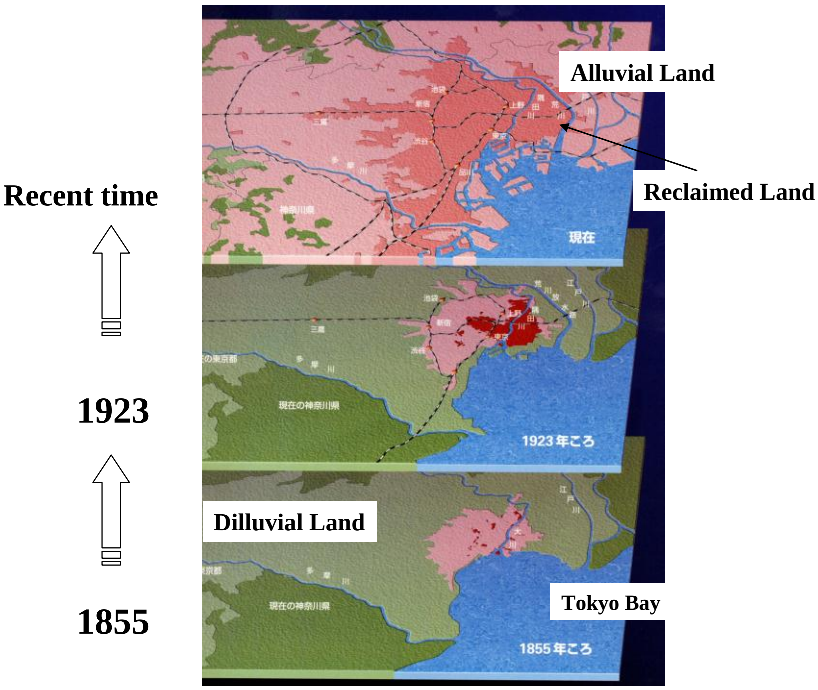
c) Lag Window


Spectrum Window, Parzen's Filter
Number of Data=2048, Nyquist Frequency=1/(2*0.01)
Band=0.0Hz
Fourier Amp, (cm/sec)
0
10
20
30
40
50
0.01
0.1
1
10
Period, (sec.)


Spectrum Window, Parzen's Filter
Number of Data=2048, Nyquist Frequency=1/(2*0.01)
Band=0.5Hz
Band=1.0Hz
Band=2.0Hz
Fourier Amp, (cm/sec)
0
10
20
30
40
0.01
0.1
1
10
Period, (sec.)



地震による建物の被害と地盤の関係: Relation between the damage of the building by the earthquake and ground condition: 「おもしろジオテク」(技報堂)



東京のウォーター・フロントの経緯(埋立て事業の経緯): Process of water-front and Reclamation in Tokyo-bay

推薦図書（Recommendation Text and Papers）

1) 「新・地震動のスペクトル解析入門」 大崎順彦，鹿島出版会
2) 「スペクトル解析」 日野幹雄，朝倉書店
3) 「フーリエ解析」 大石進一，岩波書店（理工系の数学入門コース）

For examples

1) “The Fourier Integral and Its Applications”, papoulis, A.(1962), McGraw-Hill
2) “Random Data: Analysis and measurement Procedures”, Bendat, J.S. and Piersol, A.G.(1971), John Wiley & Sons.