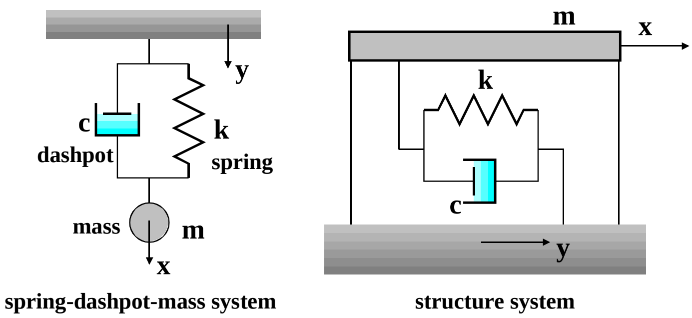
1 質点線形系の振動解析

Numerical Analysis Method for a Mass with Linear-System in 1-D



Linear Vibration System

ー１質点系の振動の数学的厳密解(***Mathematical Exact Solution***)
ー 応答の数値解法(***Numerical Solutions***)：直接積分法(***Direct Integration Method***)
ー 地震応答スペクトル（***Earthquake Response Spectrum***）

運動方程式：

$$m\ddot{x}(t)+c\dot{x}(t)+kx(t)=f(t)=-m\ddot{y}(t) \tag{1.1}$$

（貫性力：相対加速度）＋（減衰制振力：相対速度）＋（復元力：相対変位）＝（外力）

Inertia force viscous force recover force

釣合い式：

$$-m(\ddot{x}(t)+\ddot{y}(t))-c\dot{x}(t)-kx(t)=0 \tag{1.2}$$

（貫性力：絶対加速度）＋（減衰制振力：相対速度）＋（復元力：相対変位）＝0

m: mass [kg]

k: spring coefficient [kg/sec^2]

c: viscous damping coefficient [kg/sec]

$f(t)$: external force applied to the mass

$\ddot{y}(t)$: $f(t)$ is replaced with acceleration of the ceiling and the ground $-m\ddot{y}(t)$ according to D'Alembert's law.

$\ddot{x}(t)$, $\dot{x}(t)$, $x(t)$: relative acceleration[m/sec^2], velocity[m/sec], and displacement[m] with respect to the ceiling or the ground, respectively

$x(t)+y(t)$: absolute displacement

$m\ddot{x}(t)$:inertia force

1. *Mathematical Exact Solutions: Analytical results*

1.1. 自由振動（*Free Vibration*）：

$$f(t) \equiv 0\,,\quad \ddot{y}(t) \equiv 0 \tag{1.3}$$

Linear and homogeneous differential equation

solution = general solution

$$x(t) = x_g(t) \tag{1.4}$$

Characteristic Equation using differential operator

$$D^2 x_g + 2\gamma D x_g + {\omega_o}^2 x_g = 0 \tag{1.5}$$

D***: differential operator with respect to time differential (***$D = d/dt$***):*** $\dot{x}_g = Dx_g$ ***and*** $\ddot{x}_g = D^2 x_g$***.***

$$\gamma = \frac{c}{2m}\,, \qquad {\omega_o}^2 = \frac{k}{m} \tag{1.6}$$

γ : viscosity normalized by mass[1/sec]

ω_o ***: referential angular frequency[1/sec] / natural circle frequency***

$$T_o = \frac{1}{f_o} = \frac{2\pi}{\omega_o} = 2\pi\sqrt{\frac{k}{m}} \tag{1.7}$$

T_o ***: referential vibration period[sec] / natural period***

f_o ***: referential vibration frequency[1/sec] / natural frequency***

$$Dx_g = \left(\gamma \pm \sqrt{\gamma^2 - \omega_o^2}\right) x_g \tag{1.8}$$

Vibration mode of the system clearly depends on the relative magnitude of viscous damping:

$$\begin{cases} \gamma^2 - {\omega_o}^2 < 0 & \left(c < 2\sqrt{mk}\right) & \textit{normally damped vibration} \\ \gamma^2 - {\omega_o}^2 = 0 & \left(c = 2\sqrt{mk}\right) & \textit{critically damped vibration}: \quad c_{cr} = 2\sqrt{mk} \\ \gamma^2 - {\omega_o}^2 > 0 & \left(c > 2\sqrt{mk}\right) & \textit{over damped vibration} \end{cases} \tag{1.9}$$

a) *Normally damped vibration*

parameters

$$c \le c_{cr} = 2\sqrt{mk}\,, \tag{2.10}$$

$$D = (-\gamma \pm i\omega), \tag{2.11}$$

$$\omega^2 = \omega_o^2 - \gamma^2\,, \tag{2.12}$$

period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_O}\frac{1}{\sqrt{1-h^2}} = \frac{T_o}{\sqrt{1-h^2}} \tag{2.13}$$

motions

$$\begin{cases} x_g(t) = e^{-\gamma t}\left[A_g \cos\omega t + B_g \sin\omega t\right] \\ \dot{x}_g(t) = e^{-\gamma t}\left[\{-\gamma A_g + \omega B_g\}\cos\omega t + \{-\gamma B_g - \omega A_g\}\sin\omega t\right] \\ \ddot{x}_g(t) = e^{-\gamma t}\left[\{(\gamma^2 - \omega^2)A_g - 2\gamma\omega\omega B_g\}\cos\omega t + \{(\gamma^2 - \omega^2)B_g + 2\gamma\omega\omega A_g\}\sin\omega t\right] \end{cases} \tag{2.14}$$

initial conditions
$$\begin{cases} x_g(0) = A_g \\ \dot{x}_g(0) = -\gamma A_g + \omega B_g \end{cases} \tag{2.15}$$

therefore,
$$\begin{cases} A_g = x_g(0) \\ B_g = \{\dot{x}_g(0) + \gamma x_g(0)\}/\omega \end{cases} \tag{2.16}$$

complex descriptions

$$\begin{cases} x_g(t) = \mathrm{Re}\,al\left[C_g \exp\{(-\lambda - i\omega)t\}\right] \\ \dot{x}_g(t) = \mathrm{Re}\,al\left[C_g(-\lambda - i\omega)\exp\{(-\lambda - i\omega)t\}\right] \\ \ddot{x}_g(t) = \mathrm{Re}\,al\left[C_g(-\lambda - i\omega)^2 \exp\{(-\lambda - i\omega)t\}\right] \end{cases} \tag{2.17}$$

damping

Viscous damping coefficient

c

$$c_{cr} = 2\sqrt{mk} \tag{2.18}$$

Viscosity normalized by mass

$$\gamma = \frac{c}{2m}, \tag{2.19}$$

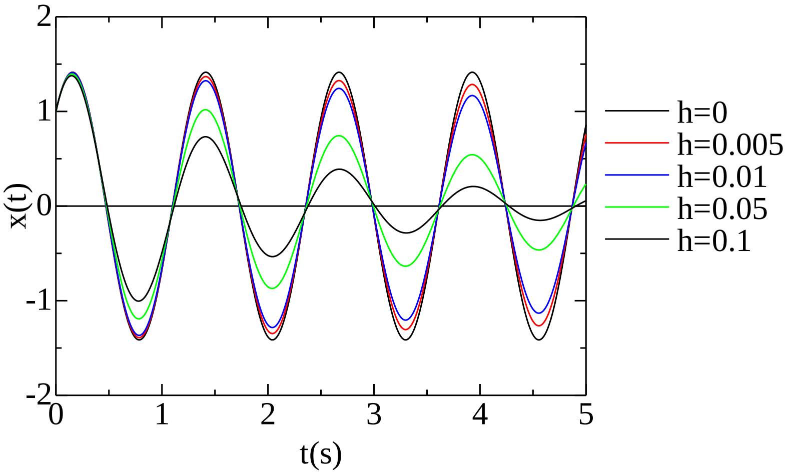
Logarithmic damping factor

$$D = \ln \frac{a_n}{a_{n+1}} \tag{2.20}$$

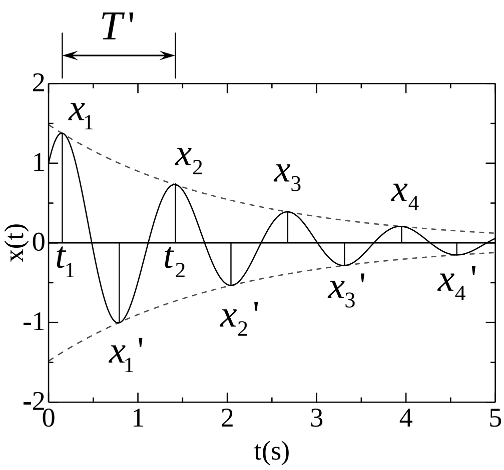
Damping factor

$$h = \frac{c}{c_{cr}} = \frac{c}{2\sqrt{mk}} = \frac{\gamma}{\omega_o} \tag{2.21}$$

$$D = \ln \frac{a_n}{a_{n+1}} = \ln \frac{x(t)}{x(t+T)} = \gamma T = \frac{2\pi h}{\sqrt{1-h^2}} \tag{2.22}$$



自由減衰振動



固有周期実測値(observed value of fundamental natural period)

structure	中低層（H≧45m）	高層（H＜45m）
S	T_1=0.061 N[階]	T_1=0.079 N[階]
SRC, RC	T_1=0.054 N[階]	T_1=0.053 N[階]

減衰定数実測値(observed value of damping factor)

鉄骨構造(Steel structure): h=2%

鉄骨鉄筋コンクリート構造(Steel reinforced concrete structure): h=3%

鉄筋コンクリート構造(Reinforced concrete structure)： h=5%

土： h=0~25%

Here, if h<< 1, we can obtain,

$$h << 1, \qquad D = 2\pi h \tag{2.23}$$

Consequently, we can estimate value of *h* by measuring *D*,

$$h << 1, \qquad h = \frac{D}{2\pi} \tag{2.24}$$

b) ***Critical damped vibration***

parameters

$$c = c_{cr} = 2\sqrt{mk}\,, \tag{2.25}$$

$$D = -\gamma\,, \tag{2.26}$$

$$\omega^2 = \omega_o^2 - \gamma^2 = 0\,, \tag{2.27}$$

period

$$T = \frac{2\pi}{\omega} \quad \Rightarrow \quad \infty \tag{2.28}$$

motions

$$\begin{cases} x_g(t) = e^{-\gamma t}\left[A_g + \gamma B_g t\right] \\ \dot{x}_g(t) = e^{-\gamma t}\left[\left(-\gamma A_g + \gamma B_g\right) + \left(-\gamma^2 B_g\ t\right)\right] \\ \ddot{x}_g(t) = e^{-\gamma t}\left[\left(\gamma^2 A_g - 2\gamma^2 B_g\right) + \gamma^3 B_g\ t\right] \end{cases} \tag{2.29}$$

initial conditions
$$\begin{cases} x_g(0) = A_g \\ \dot{x}_g(0) = -\gamma A_g + \gamma B_g \end{cases} \tag{2.30}$$

therefore,
$$\begin{cases} A_g = x_g(0) \\ B_g = \dot{x}_g(0)/\gamma + x_g(0) \end{cases} \tag{2.31}$$

c) ***Over damped vibration***

parameters

$$c \ge c_{cr} = 2\sqrt{mk}\,, \tag{2.32}$$

$$D = \left(-\gamma \pm \omega\right), \tag{2.33}$$

$$\omega^2 = -\left(\omega_o^2 - \gamma^2\right), \tag{2.34}$$

period

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_O}\frac{1}{\sqrt{h^2 - 1}} = \frac{T_o}{\sqrt{h^2 - 1}} \tag{2.35}$$

motions

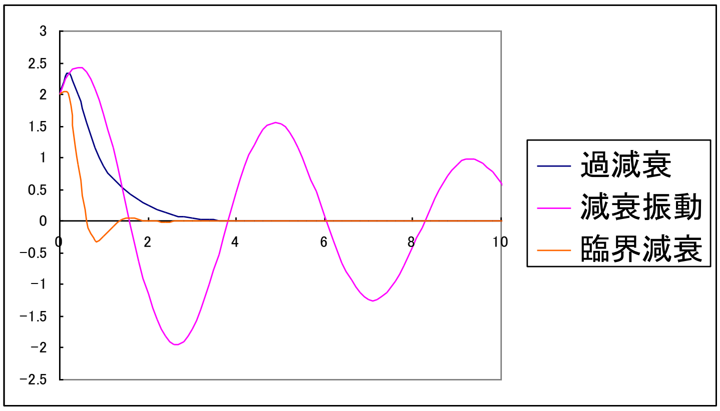
$$\begin{cases} x_g(t) = e^{-\gamma t}\left[A_g \cosh \omega t + B_g \sinh \omega t\right] \\ \dot{x}_g(t) = e^{-\gamma t}\left[\{-\gamma A_g + \omega B_g\}\cosh \omega t + \{-\gamma B_g - \omega A_g\}\sinh \omega t\right] \\ \ddot{x}_g(t) = e^{-\gamma t}\left[\{(\gamma^2 - \omega^2)A_g - 2\gamma\omega\omega B_g\}\cosh \omega t + \{(\gamma^2 - \omega^2)B_g + 2\gamma\omega\omega A_g\}\sinh \omega t\right] \end{cases} \tag{2.36}$$

initial conditions

$$\begin{cases} x_g(0) = A_g \\ \dot{x}_g(0) = -\gamma A_g + \omega B_g \end{cases} \tag{2.37}$$

therefore,

$$\begin{cases} A_g = x_g(0) \\ B_g = \{\dot{x}_g(0) + \gamma x_g(0)\}/\omega \end{cases} \tag{2.38}$$



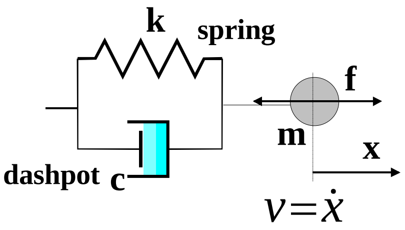
An example for vibration behaviours with different h : normally damped, critical damped and over damped vibrations

* Euler's formula

$$\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right), \ \ \sin x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$$

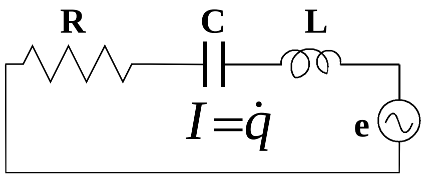
$$e^{inx} = \left(\cos x + i \sin x\right)^n = \left(\cos nx + i \sin nx\right)$$

* mechanical vibration system and electric circuit system



spring-dashpot-mass system

$$f = m\dot{v} + cv + k\int v dt$$



electric circuit system

$$e = L\dot{I} + RI + \frac{1}{C}\int I dt$$

* mechanical vibration system and a control system in control engineering: PID control

$$S_c = T_D K_p \frac{de}{dt} + K_p e + \frac{1}{T_I} K_p \int e\, dt$$

Derivation control + Proportion control + Integration control

PID

S_c : control signal

e : error = target value – current value

非線形解法 (Non-linear analysis of vibration by numerical method)

直接積分法(***Direct Integration Method***)

非線形運動方程式の数値積分の方法

⊂ 差分法

— 時間領域の数値積分

— Newmark の β 法： 実用的に最も用いられる

$$x(t+\Delta t)=x(t)+\Delta t\cdot\dot{x}(t)+\left(\frac{1}{2}-\beta\right)(\Delta t)^2\ddot{x}(t)+\beta(\Delta t)^2\cdot x(t+\Delta t) \quad (2.39)$$

$$\dot{x}(t+\Delta t)=\dot{x}(t)+(1-\gamma)\Delta t\cdot\ddot{x}(t)+\gamma\cdot\Delta t\cdot\ddot{x}(t+\Delta t) \quad (2.40)$$

Newmark の β 法の特別な呼び名

手法	γ	β
線形加速度法(Linear Acceleration Method)	0.5	1/6
中点加速度法(Constant Average Acceleration Method)	0.5	1/4

— Wilson の θ 法

$$x(t+\Delta t)=x(t)+\Delta t\cdot\dot{x}(t)+\frac{1}{2}(\Delta t)^2\ddot{x}(t)+\frac{(\Delta t)^2}{6\theta}\left(\ddot{x}(t+\theta\Delta t)-\ddot{x}(t)\right) \quad (2.41)$$

$$\dot{x}(t+\Delta t)=\dot{x}(t)+\Delta t\cdot\ddot{x}(t)+\frac{\Delta t}{2\theta}\left(\ddot{x}(t+\Delta t)-\ddot{x}(t)\right) \quad (2.42)$$

各種数値積分法の安定条件(stability condition of numerical integration)

手法	数値積分の安定条件	無条件安定条件 (unconditionally Stable)
Newmark の β 法	$\Delta t \le \dfrac{2}{\omega_o\sqrt{1-4\beta}}\,0.5$	$\beta \le 1/4$
Wilson の θ 法		$\theta \ge 1.37$
中央差分法	$\Delta t \le \dfrac{2}{\omega}$	なし

ω_o はモデルの最も大きい円振動数

一 直接積分法

ｉ）運動方程式の離散化

$$\ddot{x}(t)+2h\omega\,\dot{x}(t)+\omega^2 x(t)=-\ddot{y}(t) \tag{2.43}$$

$$\left.\begin{aligned} x_{t+\Delta t} &= A_{11}x_t + A_{12}\dot{x}_t + B_{11}\ddot{y}_t + B_{12}\ddot{y}_{t+\Delta t} \\ \dot{x}_{t+\Delta t} &= A_{21}x_t + A_{22}\dot{x}_t + B_{21}\ddot{y}_t + B_{22}\ddot{y}_{t+\Delta t} \end{aligned}\right\} \tag{2.44}$$

$$[A]=\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},\quad [B]=\begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \tag{2.45}$$

$$\begin{Bmatrix} x_{t+\Delta t} \\ \dot{x}_{t+\Delta t} \end{Bmatrix}=[A]\begin{Bmatrix} x_t \\ \dot{x}_t \end{Bmatrix}+[B]\begin{Bmatrix} \ddot{y}_t \\ \ddot{y}_{t+\Delta t} \end{Bmatrix} \tag{2.46}$$

$$\ddot{x}_{t+\Delta t}+2h\omega\,\dot{x}_{t+\Delta t}+\omega^2 x_{t+\Delta t}=-\ddot{y}_{t+\Delta t} \tag{2.47}$$

$$\ddot{x}_t+2h\omega\,\dot{x}_t+\omega^2 x_t=-\ddot{y}_t \tag{2.48}$$

$$\ddot{x}_{t+\Delta t}+2h\omega\,\dot{x}_{t+\Delta t}+\omega^2 x_{t+\Delta t}=-\ddot{y}_{t+\Delta t} \tag{2.49}$$

Taylor Expansion

$$x_{t+\Delta t}=x_t+\dot{x}_t\cdot\Delta t+\ddot{x}_{t2}\cdot\frac{(\Delta t)^2}{2} \tag{2.50}$$

$$x_{t+\Delta t}=x_t+\ddot{x}_{t1}\cdot\Delta t \tag{2.51}$$

Taylor Expansion

$$f(t+\Delta t)=f(t)+\Delta t\cdot f'(t_1)\qquad t\le t_2\le t+\Delta t \tag{2.52}$$

$$f(t+\Delta t)=f(t)+\Delta t\cdot f'(t)+f''(t_2)\frac{(\Delta t)^2}{2}\qquad t\le t_2\le t+\Delta t \tag{2.53}$$

ii）代数方程式を計算

(1) Newmark の β 法（内挿方式）

$$\ddot{x}_{t2}=(1-\gamma)\ddot{x}_t+\gamma\cdot\ddot{x}_{t+\Delta t},\quad \ddot{x}_{t1}=(1-2\beta)\ddot{x}_t+2\beta\,\ddot{x}_{t+\Delta t} \tag{2.54}$$

$$x_{t+\Delta t}=x_t+\dot{x}_t\Delta t+\left[(1-2\beta)\ddot{x}_t+2\beta\,\ddot{x}_{t+\Delta t}\right]\frac{(\Delta t)^2}{2} \tag{2.55}$$

$$\dot{x}_{t+\Delta t}=x_t+\left[(1-\gamma)\ddot{x}_t+\gamma\,\ddot{x}_{t+\Delta t}\right]\Delta t \tag{2.56}$$

$$[A]=[?]$$

$$[B]=[?]$$

(2) Wilson の θ 法（外挿方式）

時間 $t \sim t+\Delta t$ で成立する運動方程式が、時間 $t+\theta\Delta t$ $(\theta \geq 1)$ でも成立するとする。

$$\ddot{x}_{t+\theta\Delta t} + 2h\omega\,\dot{x}_{t+\theta\Delta t} + \omega^2 x_{t+\theta\Delta t} = -\ddot{y}_{t+\theta\Delta t}\,, \qquad \theta \geq 1 \tag{2.57}$$

$$\left.\begin{aligned} \ddot{x}_{t+\Delta t} &= (1-\theta)\ddot{x}_t + \theta \cdot \ddot{x}_{t+\Delta t} \\ \ddot{y}_{t+\Delta t} &= (1-\theta)\ddot{y}_t + \theta \cdot \ddot{y}_{t+\Delta t} \end{aligned}\right\} \tag{2.58}$$

$$\left.\begin{aligned} x_{t+\theta\Delta t} &= x_t + \dot{x}_t(\theta\Delta t) + \left[\left(1-\frac{\theta}{3}\right)\ddot{x} + \frac{\theta}{3}\ddot{x}_{t+\theta\Delta t}\right]\frac{(\theta\Delta t)^2}{2} \\ \dot{x}_{t+\theta\Delta t} &= \dot{x}_t + \left[\left(1-\frac{\theta}{2}\right)\ddot{x} + \frac{\theta}{2}\ddot{x}_{t+\theta\Delta t}\right](\theta\Delta t) \end{aligned}\right\} \tag{2.59}$$

In the case of θ =1,

$$\left.\begin{aligned} x_{t+\theta\Delta t} &= x_t + \dot{x}_t(\Delta t) + \left[\frac{2}{3}\ddot{x} + \frac{1}{3}\ddot{x}_{t+\theta\Delta t}\right]\frac{(\Delta t)^2}{2} \\ \dot{x}_{t+\theta\Delta t} &= \dot{x}_t + \left[\frac{1}{2}\ddot{x} + \frac{1}{2}\ddot{x}_{t+\theta\Delta t}\right](\Delta t) \end{aligned}\right\} \tag{2.60}$$

= ***Linear Acceleration Method (*** γ ***=0.5,*** β ***=1/6)***

- 地震応答スペクトル（***Earthquake Response Spectrum***）

（相対）変位応答スペクトル(***Displacement Response Spectrum***): x
（相対）速度応答スペクトル(***Velocity Response Spectrum***) : $\dot{x}$
（絶対）加速度応答スペクトル(***Acceleration Response Spectrum***) : $\ddot{x}+\ddot{y}$

$S_d(h,T)$: 最大相対変位応答値(***Displacement Response Spectrum***)
$S_v(h,T)$: 最大相対速度応答値(***Velocity Response Spectrum***)
$S_a(h,T)$: 最大絶対加速度応答値(***Acceleration Response Spectrum***)

Plots of maximum response values again selected parameters of the system or of forcing function (earthquake considered) are called 'response spectra'.

For one-degree system, the natural period (or frequency) is the characteristic that determined its response to a given forcing function

⇒ratio of maximum dynamic stress in a structure to the corresponding static stress

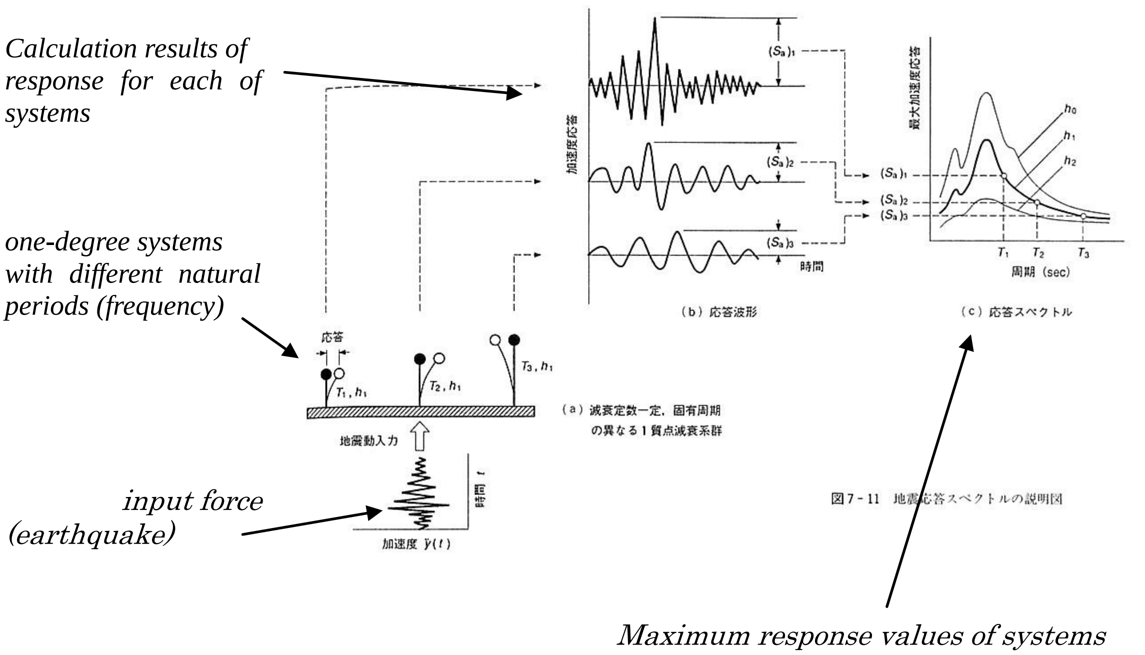


図7-11 地震応答スペクトルの説明図

$S_d \fallingdotseq \frac{1}{\omega} S_v = \frac{T}{2\pi} S_v$: 最大相対変位応答値(***Displacement Response Spectrum***)

$\boxed{S_v = S_v\text{: 最大相対速度応答値(}\textit{\textbf{Velocity Response Spectrum}}\text{)}}$

$S_a \fallingdotseq \omega S_v = \frac{2\pi}{T} S_v$: 最大絶対加速度応答値(***Acceleration Response Spectrum***)

(2.61)

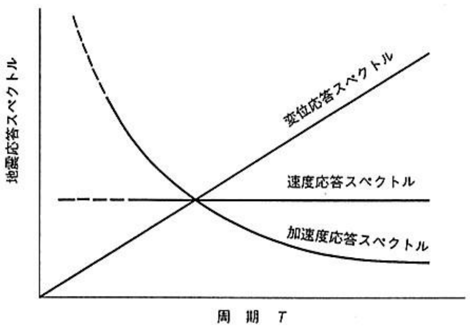


図7-16 地震応答スペクトルの概略の形

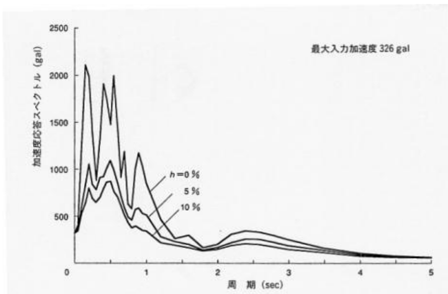


図7-12 エル・セントロ地震波の加速度応答スペクトル

S_a vs period

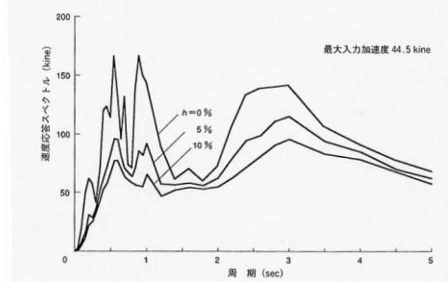


図7-13 エル・セントロ地震波の速度応答スペクトル

S_v vs period

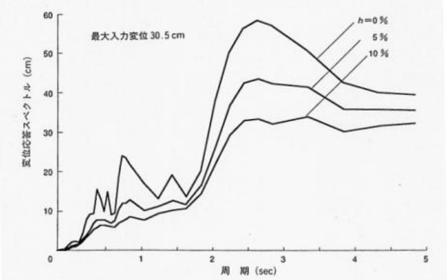


図7-14 エル・セントロ地震波の変位応答スペクトル

S_D vs period

ー 3 重応答スペクトル（***Tripartite response spectrum***）

$$\left.\begin{aligned}\log S_v &= \log S_a - \log(2\pi) + \log T \\ \log S_v &= \log S_d + \log(2\pi) - \log T\end{aligned}\right\} \tag{2.62}$$

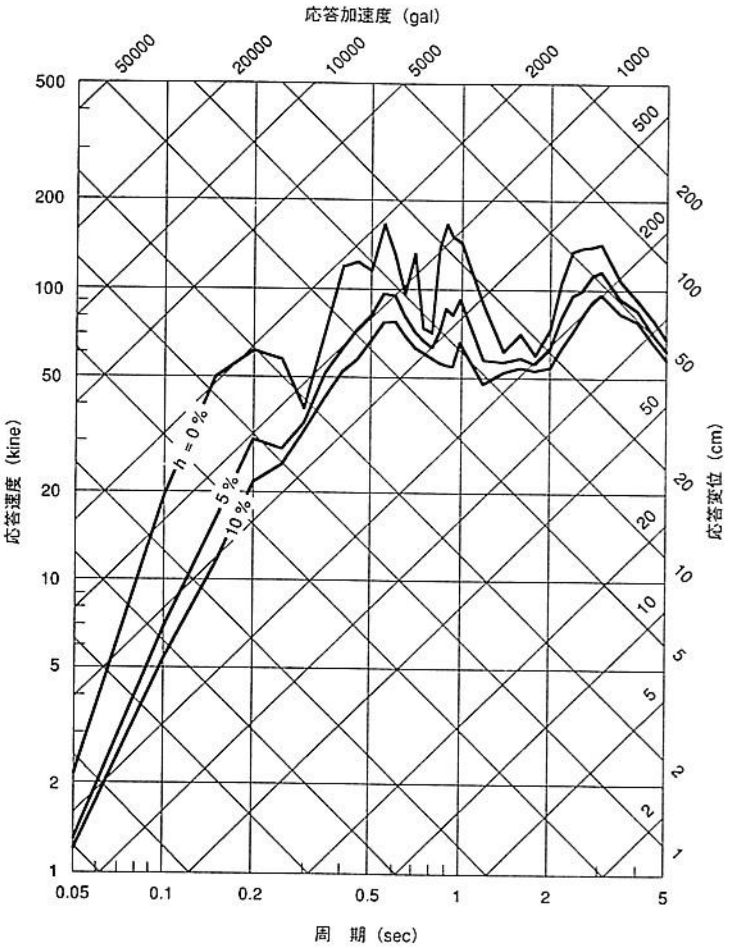


図7－19　エル・セントロ地震波の 3 重応答スペクトル

ー 応答スペクトル（*response spectrum*）について

・フーリエ・スペクトルは地震波そのものの周波数特性。応答スペクトルは構造物（1 質点減衰系）を含んだ地震動の全体像

・モード解析（modal analysis）

加速度応答スペクトル：地震力、ベース・シア係数(base shear coefficient)、動的震度

地震力

$$Q_{\max} = m(\ddot{x}+\ddot{y})_{\max} \tag{2.63}$$

ベース・シア係数(base shear coefficient)、**動的震度**

$$C = \frac{Q_{\max}}{W} = \frac{(\ddot{x}+\ddot{y})_{\max}}{g} = \frac{S_a(h,T)}{g} \tag{2.64}$$

静的震度: 静的耐震設計 k_h

$$Q_{\max} = k_h \cdot W \tag{2.65}$$

速度応答スペクトル：地震動が構造物に与える最大のエネルギー

最大ひずみエネルギー(strain energy): $\frac{1}{2}k(x_{\max})^2$ (2.66)

単位質量あたりの最大エネルギー: $\frac{1}{2}\cdot\frac{k}{m}(x_{\max})^2 = \frac{1}{2}(\omega x_{\max})^2 = \frac{1}{2}S_v^2$ (2.67)

(maximum energy per unit mass)

スペクトル強度(Spectral intensity): $I_h = \int_{0.1}^{2.5} S_v(h,T)dT$ (2.68)

変位応答スペクトル：ひずみの大きさ～応力の大きさ

最大せん断力: $Q_{\max} = k \cdot x_{\max}$ (2.69)

$$m\cdot(\ddot{x}+\ddot{y})_{\max} = kx_{\max} \tag{2.70}$$

[Recommended text]

William Weaver, Jr., Stephen P. Timoshenko, Donovan H. Young; Vibration problems in engineering, 5th edition(?), Wiley Interscience.